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# ALL UNCOUNTABLE CARDINALS CAN BE SINGULAR

BY

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ABSTRACT

Assuming the consistency of the existence of arbitrarily large strongly compact cardinals, we prove the consistency with ZF of the statement that every infinite set is a countable union of sets of smaller cardinality. Some other statements related to this one are investigated too.

## 0. Introduction

It is well known that the axiom of choice implies that a countable union of countable sets is countable and that for every ordinal $\alpha$ $\aleph_{\alpha+1}$ is a regular cardinal. Without the axiom of choice the picture is quite different.

Levy [6] proved that it is consistent with ZF that $\aleph_1$ is singular. This leads naturally to the problem of generalizing Levy's result to all $\aleph_\alpha$'s. The purpose of this paper is to show that such a generalization is possible if one is ready to allow for the consistency of some large cardinals.

The exact statement of our main theorem is:

THEOREM I. *If* ZFC + '$(\forall \alpha \in On)((\exists k > \alpha)$ *($k$ is a strongly compact cardinal))' is consistent then* ZF + '$(\forall \alpha \in On)(\mathrm{cf}\aleph_\alpha = \aleph_0)$' *is consistent too.*

It is known that the consistency of ZF + '$(\forall \alpha \in On)(\mathrm{cf}\aleph_\alpha = \aleph_0)$' cannot be proved without assuming the consistency of the existence of some large cardinals. Jensen's Covering Theorem [1] implies that if both $\aleph_1$ and $\aleph_2$ are singular then $0^{\#}$ exists. Recently Jensen and Dodd [2] were able to show that under the same assumption, one can obtain an inner model with a measurable cardinal. The large cardinals assumption which we use here is much stronger. Even to make only $\aleph_1$ and $\aleph_2$ singular we need a cardinal $k$ which is $\lambda$-strongly compact for every $\lambda < k^{+\omega}$, where $k^{+0} = k$, $k^{+(n+1)} = (k^{+n})^+$ and $k^{+\omega} = \bigcup_{n<\omega} k^{+n}$.

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Following Specker [10], let us consider the class $\Omega = \{\alpha \in 0n \mid (\forall \beta \leqq \alpha)$ $(\beta = 0 \vee (\exists \gamma)(\beta = \gamma + 1) \vee$ there is a sequence $\langle \gamma_n \mid n < \omega \rangle$ such that for each $n$ $\gamma_n < \beta$ and $\beta = \bigcup_{n<\omega} \gamma_n)$.

As Specker has shown exactly one of the following alternative hold.

(1) $\Omega = 0n$,

(2) $\Omega = \omega_\Omega$,

$(3)_\alpha$ $\Omega = \omega_{\alpha+1}$, where $\alpha$ is an ordinal.

Levy and Feferman [6] constructed a model in which $\Omega = \omega_2$. Levy [7] obtained some interesting consequences of (3). If $\Omega \supseteq \omega_3$, then Jensen and Dodd's surprising results in [2] show that there is a measurable cardinal in some inner model. Using some large cardinal assumptions, we prove that each one of the alternatives (1)–$(3)_\alpha$, for any ordinal $\alpha$, can hold. Using the same constructions we get the following:

THEOREM II. *If* ZFC + '$(\forall \alpha \in 0n)(\exists k > \alpha)$ *($k$ is a strongly compact cardinal)' is consistent, so are*

(a) ZF + '*every infinite set is a countable union of sets of smaller cardinality*'.

(b) ZF + $\Omega = 0n$ + '*there is an uncountable set $A$ such that for every sequence $\langle A_i \mid i \in I \rangle$, which is increasing with respect to inclusion, if $|A_i| < |A|$ for every $i \in I$ and $A = \bigcup_{i \in I} A_i$, then $|I| \geqq |A|$*'.

Let us give a brief outline of the proof of Theorems I and II. We first extend the universe $V$ by a filter $G$ generic over a proper class of forcing conditions which are similar to those of Prikry [8]. In the universe $V[G]$ thus obtained all regular cardinals of $V$, and hence all cardinals, become ordinals of cofinality $\omega$. $V[G]$ satisfies all the axioms of ZFC except for the power set axiom. Inside $V[G]$ we construct a symmetric model $N_G$ by the method described in Jech [4, 5]. The well ordered cardinals of $N_G$ are exactly $\omega$, all the strongly compact cardinals of $V$ and their limits. Every such cardinal has cofinality $\omega$ in $N_G$. Also, in this model every infinite set is a countable union of sets of smaller cardinality. To obtain a model which satisfies (b) of Theorem II, we add by forcing an amorphous set $A$ to this model (i.e. a set $A$ such that if $B \subseteq A$ then $|B| < \aleph_0$ or $|A - B| < \aleph_0$).

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## 1. The basic model

A cardinal $k$ is called strongly compact if every $k$-complete filter over any set $S$, $|S| \geqq k$ can be extended to a $k$-complete ultrafilter over $S$.

Let $M$ be a countable standard transitive model of ZFC + '$(\forall \alpha \in On)$ $(\exists k > \alpha)$ ($k$ is a strongly compact cardinal))'. Assume without loss of generality that there is a binary relation $W$ on $M$ which is a well ordering of $M$ and such that in the structure $\langle M, \in, W\rangle$ the axiom of replacement holds for all formulas of the language with the relation symbols $\in$ and $W$. This is a well known result the proof of which can be found, for example, in Felgner [3].

Without loss of generality we can assume that there is no regular cardinal in $M$ which is a limit of strongly compact cardinals. If such a cardinal exists, then it is inaccessible and let $\lambda$ be the least such cardinal. Consider the set $M' = \{x \in M \mid \operatorname{rank} x < \lambda\}$. $M'$ is a model of ZFC (see, Jech [4]) and has the required properties.

Let $\langle k_\alpha \mid \alpha \in On\rangle$ be the increasing sequence of all strongly compact cardinals, where $k_0 = \aleph_0$.

Definition 1.1. Let $\alpha$ be a regular cardinal, then

(1) $\alpha$ is said to be of type 1, if there is a greatest strongly compact cardinal $k \leqq \alpha$. We shall write in this case $\mathrm{cf}'\alpha = \alpha$. Let $\Phi_\alpha$ be the least $k$-complete uniform ultrafilter over $\alpha$ in the well ordering $W$ of $M$, if $k > \aleph_0$ and if $k = \aleph_0$, let $\Phi_\alpha = \{X \subseteq \alpha \mid |\alpha - X| < \alpha\}$. By $\Phi_\alpha$ being uniform we mean that if $X \in \Phi_\alpha$, then $|X| = \alpha$.

(2) $\alpha$ is of type 2, if there is no greatest strongly compact cardinal $\leqq \alpha$. Let $\gamma$ be the cofinality of the greatest cardinal $\beta \leqq \alpha$ which is a limit of strongly compact cardinals. We shall denote this $\gamma$ by $\mathrm{cf}'\alpha$.

If an $\alpha$ of type 2 is regular then, as we assumed, $\alpha$ is not a limit of strongly compact cardinals, hence $\mathrm{cf}'\alpha < \alpha$. Let $\langle k_\nu^\alpha \mid \nu < \mathrm{cf}'\alpha\rangle$ be the least ascending sequence of strongly compact cardinals $\geqq k_1$ such that $\beta = \bigcup\{k_\nu^\alpha \mid \nu < \mathrm{cf}'\alpha\}$. Denote by $\Phi_{\alpha,\nu}$ the least $k_\nu^\alpha$-complete uniform ultrafilter over $\alpha$.

## 2. The forcing conditions

For any class $R$ we denote $\{x \mid \exists y \langle y, x\rangle \in R\}$ by $\operatorname{rng} R$, $\{x \mid \exists y \langle x, y\rangle \in R\}$ by $\operatorname{dom} R$ and $\{\langle x, y\rangle \mid \exists z \langle x, \langle y, z\rangle\rangle \in R\}$ by $\operatorname{dom}^2 R$. Let $R(x) = \{y \mid \langle x, y\rangle \in R\}$, $R''(x) = \{R(y) \mid y \in x\}$, $R \mid z = \{\langle x, y\rangle \in R \mid x \in z\}$. Identify $\langle x, \langle y, z\rangle\rangle$ with

$\langle x, y, z\rangle$ and let $(\langle x, y\rangle)_1 = x$, $(\langle x, y\rangle)_2 = y$.

DEFINITION 2.1. Let $P_1$ be the class of all finite subsets $p$ of $\mathrm{Reg} \times \omega \times 0n$ (where Reg is the class of all regular cardinals), such that for every $\alpha \in \operatorname{dom} p$, $p(\alpha)$ is a 1–1 function from some finite subset of $\omega$ into $\alpha$.

We wish to change the cofinality of every regular cardinal to $\omega$. Hence to construct generically an $\omega$ sequence cofinal in such cardinal, an element of $P_1$ will be used as a partial information on such sequences. (For all $\alpha \in \operatorname{dom} p$, $p(\alpha)$ gives partial information on such a sequence.)

For $p_1, p_2 \in P_1$ we write $p_1 \approx p_2$ if $p_1 | (0n - k_1) = p_2 | (0n - k_1)$.

We shall now define $P_2$ which is a subclass of $P_1$, dense in $P_1$ in the natural partial order. The reason we restrict ourselves to $P_2$ is mainly technical.

DEFINITION 2.2. Let $P_2$ be the class of all $p$ such that:

(1) $p \in P_1$,

(2) for every $\alpha \in \operatorname{dom} p$, $\mathrm{cf}'\alpha \in \operatorname{dom} p$,

(3) for every $\alpha \in \operatorname{dom} p$, $\operatorname{dom}(p(\mathrm{cf}'\alpha)) \supseteq \operatorname{dom}(p(\alpha))$,

(4) let $\{\alpha_0, \cdots, \alpha_{n-1}\}$ be the increasing enumeration of $\operatorname{dom} p - k_1$. There are $m, j \in \omega$, $m \geqq 1$, $j \leqq n-1$ such that $\operatorname{dom}(p(\alpha_k)) = m+1$ for every $k < j$ and $\operatorname{dom}(p(\alpha_k)) = m$ for every $j \leqq k < n$. This $m$ and $j$ are obviously unique for $p$, therefore we put $m(p) = m$, $\alpha(p) = \alpha_j$. ($\langle \alpha(q), m(q)\rangle$ is the point we have to fill first above $k_1$ if we want to extend $q$.)

As a result of requirement (3) and (4), we see that for $p \in P_2$ there is $q \in P_2$ such that $\operatorname{dom}^2 q = \operatorname{dom}^2 p \cup \{\langle \alpha(p), m(p)\rangle\}$ iff $\mathrm{cf}'\alpha(p) = \alpha(p)$ or $\mathrm{cf}'\alpha(p) < \alpha(p)$ and $\langle \mathrm{cf}'\alpha(p), m(p)\rangle \in \operatorname{dom}^2 p$. We call the member $p$ of $P_2$ with this property extendable. Notice that by (4) if $\mathrm{cf}'\alpha(p) \geqq k_1$ then $p$ is extendable. Using $P_2$ (or equivalently $P_1$) as the set of forcing conditions, will generate for each $\alpha \in \mathrm{Reg}$ a collapsing function from $\omega$ onto $\alpha$. We have to be more careful. Prikry's conditions for changing the cofinality of a measurable cardinal to $\omega$ attend this problem by attaching to each condition an element $A \subseteq k$. $A$ is a member of a fixed normal ultrafilter on $k$. The new conditions have the form $\langle p, A\rangle$ where $p$ is, as before, partial information on $\omega$ sequence cofinal in $k$ and $A$ is the set of possible candidates for extending the sequence. The fact that the ultrafilter is $k$-complete is used heavily in the proof that no cardinals are collapsed.

We can give up the condition of normality in the Prikry conditions if we allow $A$ to be a set of possible extensions of $P$. The condition $A \in U$ is replaced by a condition requiring that for every $q \in A$ we have "many" possibilities of extending $q$ in $A$, namely $\{\beta \mid q^\frown\{\beta\} \in A\} \in U$.

(Compare condition P5, P8, P9 below in Definition 2.3.)

Now suppose we want to change cofinality of *every* regular cardinal between $k_f$ and $k_{f+1}$. We can take some kind of product of Prikry type conditions, where for each $\alpha \in \mathrm{Reg} \cap (k_{\alpha+1} - k_\alpha)$ we use for $U$ $\Phi_\alpha$ which is a uniform $k_\gamma$-complete ultrafilter on $\alpha$.

The $k_\gamma$-completeness is used to show that there was too much damage done below $k_\alpha$. For attacking every regular cardinal we have to take some kind of product of these forcings, using for each $\alpha$ the last strongly compact cardinal (if there is such, namely $\mathrm{cf}'(\alpha) = \alpha$).

$\alpha$ such that $\mathrm{cf}'(\alpha) < \alpha$ present a special problem, since we cannot find an ultrafilter on $\alpha$ which will be complete enough so that not too many cardinals below $\alpha$ are collapsed. The method of changing cofinalities of such cardinals without collapsing cardinals below was suggested to us by Menachem Magidor. We take care of such $\alpha$'s by picking a sequence of uniform ultrafilters on $\alpha$ (namely $\Phi_{\alpha,\nu}$) each of them having larger and larger degree of completeness. Namely, if $p(\alpha)$ is going to describe the $\omega$ sequence cofinal in $\alpha$, and $p(\alpha)(n)$ is defined yet we want the set of possible candidates for $p(\alpha)(n)$ to be in $\Phi_{\alpha,\nu_n}$ for some $\nu_n$, we want the $\nu_n$ to be cofinal in $\mathrm{cf}'(\alpha)$, so that essentially for $n$ large enough any degree of completeness below $\mathrm{Sup}\{k^\alpha_\nu \mid \nu < \mathrm{cf}'(\alpha)\}$ will be achieved.

That is a little bit difficult to get since $\mathrm{cf}'(\alpha)$ may be greater than $\omega$, but we do try to make it of cofinality $\omega$ as well, so we do have partial information on an $\omega$ sequence cofinal in $\mathrm{cf}'(\alpha)$ and we are going to use this information by picking $\nu_n$ to be the $n$th member of the sequence cofinal in $\mathrm{cf}'(\alpha)$. (See condition P9.)

That explains why we wanted in Definition 2.2 condition (3) that $\mathrm{dom}(p(\mathrm{cf}'\alpha)) \supseteq \mathrm{dom}(p(\alpha))$ because before we go on with the sequence we want to know the sequence for $\mathrm{cf}'(\alpha)$ at least for that length.

Definition 2.3. A member of $P_3$ is a pair $\langle p, U\rangle$ which satisfies requirements (P1)–(P10) below.

(P1) $p \in P_2$.

(P2) $U$ is a subset of $P_2$.

(P3) $p \in U$.

Before listing the additional requirements, let us explain the meaning of the conditions. $\langle p, U\rangle$ is a condition similar to those used by Prikry [8]. $\langle p, U\rangle$ is such that $p$ gives partial information on $G$, and $U$ restricts the choices of additional information to members of $U$.

(P4) For every $q \in U$, $q \supseteq p$ and $\operatorname{dom} q = \operatorname{dom} p$.

(P5) For every $q \in U$ such that $q \approx p$, if $\langle \alpha, m \rangle \in (k_1 \cap \operatorname{dom} p) \times \omega - \operatorname{dom}^2 q$ then $\{\beta \mid q \cup \langle \alpha, m, \beta \rangle\} \in U\} \in \Phi_\alpha$.

(P6) For every $q_1, q_2 \in U$ if $q_2 \approx p$ and $q_1 \cup q_2 \in P_1$ then $q_1 \cup q_2 \in U$.

(P7) For every $q \in U$ and $a \subseteq k_1 \times \omega \times k_1$, $p \cup (q \cap a) \in U$.

(P8) For every $q \in U$ if $\mathrm{cf}'(q) = \alpha(q)$ (where $\alpha(q)$ is defined in Definition 2.2 (4)) then $\{\beta \mid q \cup \{\langle \alpha(q), m(q), \beta \rangle\} \in U\} \in \Phi_{\alpha(q)}$.

(P9) For every $q \in U$ if $\mathrm{cf}'\alpha(q) < \alpha(q)$ and $m(q) \in \operatorname{dom}(q(\mathrm{cf}'\alpha(q)))$, $\{\beta \mid q \cup \{\langle \alpha(q), m(q), \beta \rangle\} \in U\} \in \Phi_{\alpha(q), q(\mathrm{cf}'\alpha(q))(m(q))}$.

(P10) If $q \in U$ and $q \not\approx p$ then there is a unique triple $\langle \alpha, m, \beta \rangle$ such that $q' = q - \{\langle \alpha, m, \beta \rangle\} \in U$ and $\langle \alpha, m \rangle = \langle \alpha(q'), m(q') \rangle$. We shall denote this $q'$ by $q^-$.

An extension of $\langle p, U \rangle$ which does not change $\operatorname{dom}(p)$ will be of the form $\langle q, V \rangle$ where $q \in U$ and $V \subseteq U$. Conditions (P5)–(P10) guarantee that rich enough collection of $q$'s in $U$. For instance, (P5) claims that practically any additional requirements we want to add to $p$ and which refer to cardinals below $k_1$ can be added to $p$ and shall keep us in $U$.

(P6) claims that if we extended $p$ as above just by information below $k_1$, then this information can be added to any other condition in $u$ provided it is consistent with this condition.

(P8) and (P9) deal with extensions above $k_1$.

If $q \in U$ we know that $(\alpha(q), m(q))$ is the first point that has to be filled above $k_1$. (P8) and (P9) claim that there are many possibilities to extend $q$ to $(\alpha(q), m(q))$. Namely, the set of these possibilities is the appropriate ultrafilter.

(P7) and (P10) claim that $U$ is closed under certain restrictions and thus certain "holes" in $U$ are avoided.

From now on whenever $q \in P_2$ is extendible, we shall denote the appropriate ultrafilter given by (P8) or (P9) with $\Phi_q$, i.e., if $\mathrm{cf}'\alpha(q) = \alpha(q)$ then $\Phi_q$ is $\Phi_{\alpha(q)}$ and if $\mathrm{cf}'\alpha(q) < \alpha(q)$, then $\Phi_q$ is $\Phi_{\alpha(q), a(\mathrm{cf}'\alpha(q))(m(q))}$.

Lemma 2.4. *If $\langle p, U \rangle \in P_3$, then $\langle p \mid a, \{t \mid a \mid t \in U\} \in P_3$ for every set $a$ of regular cardinals such that if $\alpha \in a$ then $\mathrm{cf}'\alpha \in a$.*

Proof. Conditions (P1)–(P4) hold trivially for $\langle p \mid a, \{t \mid a \mid t \in U\} \rangle$. (P5) also holds since if $q = t \mid a$ for some $t \in U$ and $q \approx p \mid a$ then since (P7) holds for $\langle p, U \rangle$ we can choose a $t$ such that $t \approx p$. Then, since (P5) holds for $\langle p, U \rangle$, we obtain that $\{\beta < \alpha \mid t \cup \{\langle \alpha, m \rangle\} \in U\} \in \Phi_\alpha$ for every pair $\langle \alpha, m \rangle \in (k_1 \cap \operatorname{dom}(p \mid a)) \times \omega - \operatorname{dom}^2 q$. Hence

$$\{\beta < \alpha \mid q \cup \{\langle \alpha, m, \beta \rangle\} \in \{t' \mid a \mid t' \in U\}\} \in \Phi_\alpha.$$

Now we want to prove (P6). Let $q_1 = t_1 \mid a$, $q_2 = t_2 \mid a$ where $t_1, t_2 \in U$ and $t_2 \approx p$. Since (P7) holds for $\langle p, U \rangle$, $p \cup t_2 \mid a \in U$. If $q_1 \cup q_2 \in P_1$, then $t_1 \cup p \cup (t_2 \mid a) \in P_1$ and hence, since (P6) holds for $\langle p, U \rangle$, $t_1 \cup p \cup (t_2 \mid a) \in U$. $(t_1 \cup p \cup (t_2 \mid a)) \mid a = t_1 \mid a \cup t_2 \mid a = q_1 \cup q_2$, i.e., $q_1 \cup q_2 \in \{t' \mid a \mid t' \in U\}$.

Conditions (P7)–(P9) follow easily from the assumption that $\langle p, U \rangle \in P_3$.

Let us prove (P10). Let $q = t \mid a$ for some $t \in U$. We can clearly assume that $t$ is of minimal cardinality among those $t \in U$ for which $q = t \mid a$. If $q \not\approx p \mid a$ then, obviously, $t \not\approx p$, hence $t^- \in U$. By the minimality of $|t|$, $t^- \mid a \not\approx q$ and hence, as easily seen, $\alpha(t^-) \in a$ and $q^- = t^- \mid a$. Thus $q^- \in \{t \mid a \mid t \in U\}$, which completes our proof. □

Lemma 2.5. *If $\langle p, U \rangle \in P_3$, $t \in U$, then $\langle t, U_t \rangle \in P_3$, where $U_t = \{q \in U \mid q \supseteq t\}$.*

Proof. Conditions (P1)–(P4), (P8), (P9) hold trivially. Let us prove (P5). Let $\langle \alpha, m \rangle \in (k_1 \cap \operatorname{dom} t) \times \omega - \operatorname{dom}^2 q$, for some $q \in U_t$, $q \approx t$. Let $q' = q \mid k_1 \cup p$. Then since $\langle p, U \rangle$ satisfies (P5) we obtain that $A = \{\beta \mid q' \cup \{\langle \alpha, m, \beta \rangle\} \in U\} \in \Phi_\alpha$. But (P6) implies that for every $\beta \in A$, $q \cup q' \cup \{\langle \alpha, m, \beta \rangle\} \in U$ hence $q \cup \{\langle \alpha, m, \beta \rangle\} \in U_t$ for every $\beta \in A \in \Phi_\alpha$.

Let us prove now that $\langle t, U_t \rangle$ satisfies (P6). Let $q_1, q_2 \in U_t$, $q_2 \approx t$ and $q_1 \cup q_2 \in P_1$. Let $q_2' = q_2 \mid k_1 \cup p$. Since $\langle p, U \rangle$ satisfies (P7) $q_2' \in U$. Clearly $q_1 \cup q_2' = q_1 \cup q_2 \in P_2$, hence since $q_2' \approx p$ and $\langle p, U \rangle$ satisfies (P6), $q_1 \cup q_2 = q_1 \cup q_2' \in U$.

To get (P7), let $q \in U_t$, $a \subseteq k_1 \times \omega \times k_1$. since $\langle p, U \rangle$ satisfies (P7) $p \cup (q \cap a) \in U$. Notice that $t \cup (p \cup (q \cap a)) \in P_1$ since $q \supseteq t \supseteq p$. Since $\langle p, U \rangle$ satisfies (P6) and $p \cup (q \cap a) \approx p$ we have

$$t \cup (q \cap a) = t \cup (p \cup (q \cap a)) \in U,$$

since $t \cup (q \cap a) \supseteq t$, $t \cup (q \cap a) \in U_t$.

To see that $\langle t, U_t \rangle$ satisfies (P10) all we have to do is to notice that if $q \in U_t$ and $q \not\approx t$ then also $q^- \supseteq t$, but it follows immediately from (P4) of Definition 2.2. □

Definition 2.6. For $p \in P_2$ and $b \supseteq \operatorname{dom} p$ such that $b$ is closed under cf′ and $b \subseteq \operatorname{Reg}$, $p' \in P_2$ is called a $b$-extension of $p$ if $\operatorname{dom} p' = b$ and $p' \mid \operatorname{dom} p = p$.

Lemma 2.7. *Let $\langle p, U \rangle \in P_3$, $b \supseteq \operatorname{dom} p$, and $b \subseteq \operatorname{Reg}$. Let $p'$ be a $b$-extension of $p$ and $U' = \{q' \in P_2 \mid q' \supseteq p'$ and $q'$ is a $b$-extension of some $q \in U\}$. Then $\langle p', U' \rangle \in P_3$.*

This lemma follows directly from the definition of $P_3$.

LEMMA 2.8. *If* $\langle p, U\rangle, \langle p, V\rangle \in P_3$ *then also* $\langle p, U \cap V\rangle \in P_3$.

PROOF. Note that for every $q \in U \cap V$, if we can extend $q$ on a pair $\langle \alpha, m\rangle$ in $U$ (i.e., $\langle \alpha, m\rangle \in \operatorname{dom} p \times \omega - \operatorname{dom}^2 q$ and for some $\beta < \alpha$, $q \cup \{\langle \alpha, m, \beta\rangle\} \in U$), then we can extend $q$ on $\langle \alpha, m\rangle$ in $V$. The set of such extensions for $U$ and for $V$ is in the filter over $\alpha$, so also the $U \cap V$ set of extensions of $q$ lies in this filter. This ensures that $\langle p, U \cap V\rangle$ is a condition. $\square$

DEFINITION 2.9. For $\langle p, U\rangle, \langle q, V\rangle \in P_3$ we put $\langle q, V\rangle \geqq \langle p, U\rangle$ and say that $\langle q, V\rangle$ is stronger than $\langle p, U\rangle$ and also that $\langle q, V\rangle$ extends $\langle p, U\rangle$ if $V \mid \operatorname{dom} p \subseteq U$, where $V \mid \operatorname{dom} p$ denotes $\{t \mid \operatorname{dom} p \mid t \in V\}$.

## 3. The main model

DEFINITION 3.1. Let $G$ be a group of permutations of $\mathrm{Reg} \times \omega \times On$. For $\pi \in G$

(1) $|\operatorname{dom}(\operatorname{dom} \pi)| < \aleph_0$,

(2) for every $\alpha \in \operatorname{dom}\operatorname{dom} \pi$ there is a permutation $\pi^\alpha$ which is defined on some finite subset of $\alpha$ such that for every $\beta < \alpha$, $n < \omega$,

$$\pi(\langle \alpha, n, \beta\rangle) = \langle \alpha, n, \pi^\alpha(\beta)\rangle \quad \text{if } \beta \in \operatorname{dom} \pi^\alpha,$$

$$\pi(\langle \alpha, n, \beta\rangle) = \langle \alpha, n, \beta\rangle \quad \text{if } \beta \notin \operatorname{dom} \pi^\alpha.$$

If $a$ is a finite set of regular cardinals then let $H_a = \{\pi \in G \mid \forall \alpha \in a, \pi^\alpha$ is the identity function on $\operatorname{dom} \pi^\alpha$, if $\alpha \in \operatorname{dom}(\operatorname{dom} \pi)\}$. Obviously $H_a$ is a normal subgroup of $G$. For $\pi \in G$ let $P^\pi$ be the subclass of $P_3$ such that $\langle p, U\rangle \in P^\pi$ if conditions (1)–(3) below hold:

(1) $\operatorname{dom} p \supseteq \operatorname{dom}(\operatorname{dom} \pi)$,

(2) for every $\alpha \in \operatorname{dom} p$, $\operatorname{dom}(p(\alpha)) = \operatorname{dom}(p(\operatorname{cf}' \alpha))$,

(3) for every $\alpha \in \operatorname{dom}(\operatorname{dom} \pi)$, $\operatorname{rng}(p(\alpha)) \supseteq \{\beta \in \operatorname{dom} \pi^\alpha \mid \exists q \in U, \beta \in \operatorname{rng}(q(\alpha))\}$. Notice that not always does $\operatorname{rng}(p(\alpha)) \supseteq \operatorname{dom} \pi^\alpha$.

It can be easily shown, using Lemmas 2.5 and 2.7, that $P^\pi$ is a dense subclass of $\langle P_3, \geqq\rangle$.

For $\langle p, U\rangle \in P^\pi$ define $\pi(\langle p, U\rangle)$ to be

$$\langle \pi p, \pi U\rangle = \langle (\pi''(p \mid \operatorname{dom}(\operatorname{dom} \pi))) \cup (p - p \mid \operatorname{dom}(\operatorname{dom} \pi)),$$

$$\{(\pi''(t \mid \operatorname{dom}(\operatorname{dom} \pi))) \cup (t - t \mid \operatorname{dom}(\operatorname{dom} \pi)) \mid t \in U\}\rangle.$$

LEMMA 3.2. *The mapping $\pi$ of $P^\pi$ just defined is an automorphism of* $\langle P^\pi, \geqq \rangle$.

PROOF. As easily seen, it is enough to prove that (P9) holds for $\pi(\langle p, U\rangle)$. Let $q \in U$ and $\mathrm{cf}'(\alpha(q)) < \alpha(q)$. If $\gamma = q\ (\mathrm{cf}'(\alpha(q))(m(q)) \in \mathrm{dom}\, \pi^{\mathrm{cf}'(\alpha(q))}$ then by (3) above $\gamma \in \mathrm{rng}(p(\mathrm{cf}'(\alpha(q))))$ and since $q \supseteq p$ and $q(\mathrm{cf}'(\alpha(q)))$ is a 1–1 function $m(q) \in \mathrm{dom}(p(\mathrm{cf}'(\alpha(q))))$. By (2) above $m(q) \in \mathrm{dom}(p(\alpha(q)))$, hence $m(q) \in \mathrm{dom}(q(\alpha(q))$, but this contradicts the definitions of $m(q)$ and $\alpha(q)$. Therefore $\gamma \notin \mathrm{dom}\, \pi^{\mathrm{cf}'\alpha(q)}$. Hence $\{\beta \mid \pi q \cup \{\langle \alpha(\pi q), m(\pi q), \beta\rangle \in \pi U\} \supseteq \{\beta \mid q \cup \{\langle \alpha(q), m(q), \beta\rangle\} \in U\} - \mathrm{dom}\, \pi^{\alpha(q)} \in \Phi_{\alpha(q),\gamma}$, since $\alpha(\pi q) = \alpha(q)$ and $m(\pi q) = m(q)$. Also since $\gamma \notin \mathrm{dom}\, \pi^{\mathrm{cf}'\alpha(q)}$, $\gamma = (\pi q)(\mathrm{cf}'(\alpha(\pi q)))(m(\pi q))$. □

Notice that, since $P^\pi$ is a dense subclass of $P_3$, $\pi$ can be extended to a unique automorphism of the complete Boolean algebra $B = \mathrm{RO}(P_3)$ (for the definition of RO see Jech [4]). We denote the extended automorphism with the same letter.

Let $G$ be a generic subclass of $P_3$ and $N_G$ the symmetric submodel of $M[G]$ which is generated by $\{H_a \mid a$ is a finite set of regular cardinals$\}$. The construction of the symmetric models is given in Jech's books [4, 5]. For any $x \in M[G]$ we shall denote by $\boldsymbol{x}$ a name of $x$ in $M$, i.e. an element $a$ of $M$ such that $K_G(a) = x$, where $K_G$ is as in Shoenfield [9]. For $\alpha \in \mathrm{Reg}$ let $P_\alpha = \{\langle p, U\rangle \in P_3 \mid \mathrm{dom}\, p \subseteq \alpha\}$ and $G_\alpha = G \cap P_\alpha$. Then $G_\alpha$ is a generic subset of $P_\alpha$ and by Shoenfield [9], $M[G] = \bigcup\{M[G_\alpha] \mid \alpha \in \mathrm{Reg}\}$. It can be easily seen that $M[G]$ satisfies the axioms of extensionality, pairing, union and infinity. Also notice that every $M[G_\alpha]$ is a model of ZFC.

LEMMA 3.3. *Let $\Phi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r)$ be a formula, $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r \in$ HS (HS is the class of all hereditarily symmetric names* — see Jech [4, 5]) *and let a be a finite set of regular cardinals such that* $\mathrm{sym}(\langle \boldsymbol{x}_1, \cdots, \boldsymbol{x}_r\rangle) \supseteq H_a$, *where* $\mathrm{sym}\, \boldsymbol{x} = \{\pi \in \boldsymbol{G} \mid \pi \boldsymbol{x} = \boldsymbol{x}\}$. *Assume that for every $\alpha \in a$ $\mathrm{cf}'\alpha \in a$.*

*If $\langle p, U\rangle \Vdash \Phi(x_1, \cdots, x_r)$, then $\langle p \mid a, U \mid a\rangle \Vdash \Phi(x_1, \cdots, x_r)$.*

PROOF. Suppose that $\langle p \mid a, U \mid a\rangle \nVdash \Phi(x_1, \cdots, x_r)$. Then there is a condition $\langle q, V\rangle$ stronger than $\langle p \mid a, U \mid a\rangle$ which forces $\Phi(x_1, \cdots, x_r)$. It is now enough to show that there are conditions $\langle p', U'\rangle$, $\langle q', V'\rangle$ and an automorphism $\pi \in H_a$ such that $\langle p', U'\rangle \geqq \langle p, U\rangle$, $\langle q', V'\rangle \geqq \langle q, V\rangle$ and $\pi(\langle p', U'\rangle) = \langle q', V'\rangle$; this will yield the required contradiction since we get $\langle p', U'\rangle \leqq_B \|\Phi(x_1, \cdots, x_r)\|^B = \|\Phi(\pi x_1, \cdots, \pi x_r)\|^B\ {}_B\!\geqq \langle q', V'\rangle \leqq_B - \|\Phi(x_1, \cdots, x_r)\|^B$ where $\leqq_B$ is the partial ordering of the Boolean algebra $B = \mathrm{RO}(P_3)$.

We shall now extend $\langle p, U\rangle$ and $\langle q, V\rangle$ in several steps. First we shall extend $\langle p, U\rangle$ to $\langle p^*, U^*\rangle$ such that $p^* \mid a = q \mid a$. Since $\langle p \mid a, U \mid a\rangle \leqq \langle q; V\rangle$ we get that $q \mid a \in U \mid a$ and hence $q \mid a = t \mid a$ for some $t \in U$. Let $\langle p^*, U^*\rangle$ be $\langle t, \{s \in U \mid s \supseteq t\}\rangle$. It is condition by Lemma 2.5, $\langle p^*, U^*\rangle \geqq \langle p, U\rangle$ and $p^* \mid a = q \mid a$. Using Lemma 2.7 we can extend $\langle p^*, U^*\rangle$ and $\langle q, V\rangle$ to $\langle p_1, U_1\rangle$ and $\langle q_1, V_1\rangle$ respectively, such that $\operatorname{dom} p_1 = \operatorname{dom} q_1$ and $p_1 \mid a = q_1 \mid a$. It can be easily seen that we can now extend $\langle p_1, U_1\rangle$, $\langle q_1, V_1\rangle$ to the respective conditions $\langle p_2, U_2\rangle$, $\langle q_2, V_2\rangle$ so that $\operatorname{dom} p_2 = \operatorname{dom} q_2$, for every $\alpha \in \operatorname{dom} p_2$

$$\operatorname{dom}(p_2(\alpha)) = \operatorname{dom}(q_2(\alpha)) = \operatorname{dom}(p_2(\operatorname{cf}'(\alpha))) = \operatorname{dom}(q_2(\operatorname{cf}'(\alpha))),$$

and $p_2 \mid a = q_2 \mid a$. Let

$$U_3 = \{t \in U_2 \mid \forall \alpha \in \operatorname{dom} p_2 - a, (\operatorname{rng} t(\alpha) - \operatorname{rng} p_2(\alpha)) \cap \operatorname{rng} q_2(\alpha) = \varnothing\},$$

$$V_3 = \{t \in V_2 \mid \forall \alpha \in \operatorname{dom} q_2 - a, (\operatorname{rng} t(\alpha) - \operatorname{rng} q_2(\alpha)) \cap \operatorname{rng} p_2(\alpha) = \varnothing\}.$$

Now we are ready to define the automorphism. For $\beta, \gamma \in On$ and $\alpha \in \operatorname{dom} p_2 - a$ let $\rho^\alpha(\beta) = \gamma$ if for some $n < \omega$, $p_2(\alpha)(n) = \beta$ and $q_2(\alpha)(n) = \gamma$. Let $f^\alpha$ be a 1–1 correspondence between $(\operatorname{rng} \rho^\alpha - \operatorname{dom} \rho^\alpha)$ and $(\operatorname{dom} \rho^\alpha - \operatorname{rng} \rho^\alpha)$. Denote by $\pi^\alpha$ the finite permutation of $\alpha$ which is defined as follows: $\operatorname{dom} \pi^\alpha = \operatorname{dom} \rho^\alpha \cup \operatorname{rng} \rho^\alpha$, for $\gamma \in \operatorname{dom} \pi^\alpha$ let $\pi^\alpha(\gamma) = \rho^\alpha(\gamma)$ if $\gamma \in \operatorname{dom} \rho^\alpha$ and $\pi^\alpha(\gamma) = f^\alpha(\gamma)$ otherwise. Let $\pi$ be the member of $G$ obtained from these $\pi^\alpha$'s, for $\alpha \in \operatorname{dom} p_2 - a$, in the obvious way. First note that $\langle p_2, U_3\rangle, \langle q_2, V_3\rangle \in P^\pi$. Let $\alpha \in \operatorname{dom} \operatorname{dom} \pi$, then $\alpha \in \operatorname{dom} p_2 - a$ and if for some ordinal $\beta \in \operatorname{dom} \pi^\alpha = \operatorname{rng} p_2(\alpha) \cup \operatorname{rng} q_2(\alpha)$ and $t \in U_3$, $\beta \in \operatorname{rng} t(\alpha)$, then by the choice of $U_3$, $\beta \in \operatorname{rng} p_2(\alpha)$. Hence we can apply $\pi$ to $\langle p_2, U_3\rangle$ and $\langle q_2, V_3\rangle$. Let $V' = V_3 \cap \pi U_3$ and $U' = \{\pi^{-1}(t) \mid t \in V'\}$. Since $\pi(p_2) = q_2$ we have, by Lemma 3.2, $\langle q_2, \pi U_3\rangle \in P^\pi$, and it follows easily that also $\langle q_2, V'\rangle \in P^\pi$. Applying Lemma 3.2 again (for $\pi^{-1}$) we get $\langle p_2, U'\rangle \in P^\pi$. So we obtained respective extensions $\langle p_2, U'\rangle$ and $\langle q_2, V'\rangle$ of $\langle p, U\rangle$ and $\langle q, V\rangle$ such that for $\pi \in H_a$, $\pi(\langle p_2, U'\rangle) = \langle q_2, V'\rangle$. □

Lemma 3.4. *Every limit ordinal $\alpha$ is of cofinality $\aleph_0$ in $N_G$.*

Proof. It is enough to prove the lemma only for every regular cardinal $\alpha$ in $M$. For such an $\alpha$ and for every $\tau < \alpha$, the class $D_\tau = \{\langle p, U\rangle \in P_3 \mid \exists n < \omega, p(\alpha)(n) \geqq \tau\}$ is a dense subclass of $\langle P_3, \geqq\rangle$. Hence $G \mid \{\alpha\}$ is a function from $\omega$ on an unbounded subset of $\alpha$. The natural name of this function is symmetric since it is invariant under $H_{\{\alpha\}}$. Hence $G \mid \{\alpha\} \in N_G$. □

## 4. The axiom of replacement

Let $A(x_1, x_2)$, $B(x_1)$ be new predicates, where $B(x_1)$ will assert that $x_1 \in M$ and $A(x_1, x_2)$ will assert that $\langle x_1, x_2 \rangle$ is in the generic filter $G$. We recall that $W$ is a binary relation which well orders $M$.

DEFINITION 4.1. Let $t \in P_3$, then we define:

(1) $t \Vdash B(\boldsymbol{x})$, if $(\forall t' \geqq t)\ (\exists t'' \geqq t')\ (\exists y \in M)\ (t'' \Vdash \check{y} = \boldsymbol{x})$,

(2) $t \Vdash A(\boldsymbol{x}_1, \boldsymbol{x}_2)$, if $(\forall t' \geqq t)\ (\exists t'' \geqq t')\ (\exists y \in P_1)\ ((\exists y_2 \in M)\ (y_2 \subseteq P_2))$ $(\langle y_1, y_2 \rangle \in P_3)$ and $(t'' \Vdash (\boldsymbol{x}_1 = \check{y}_1)$ and $(\boldsymbol{x}_2 = \check{y}_2))$ and $t'' \geqq \langle y_1, y_2 \rangle$,

(3) $t \Vdash W(\boldsymbol{x}_1, \boldsymbol{x}_2)$, if $(\forall t' \geqq t)\ (\exists t'' \geqq t') \exists y_1 \exists y_2\ (W(y_1, y_2)$ and $t'' \Vdash \check{y}_1 = \boldsymbol{x}_1$ and $t'' \Vdash \check{y}_2 = \boldsymbol{x}_2)$.

DEFINITION 4.2.

(1) $M[G] \vDash B(x)$ if $x \in M$,

(2) $M[G] \vDash A(x_1, x_2)$ if $\langle x_1, x_2 \rangle \in G$,

(3) $M[G] \vDash W(x_1, x_2)$ if $x_1, x_2 \in M$ and $M \vDash W(x_1, x_2)$.

LEMMA 4.3. *Let $\Phi(x_1, \cdots, x_r)$ be a formula of the language of $\langle \in, =, A, B, W \rangle$, then $M[G] \vDash \varphi(x_1, \cdots, x_r)$ iff there is a $t \in G$ $t \Vdash \varphi(x_1, \cdots, x_r)$.*

The proof is similar to the proofs of the analogous lemmas in Shoenfield [9].

LEMMA 4.4. *There is definable well ordering of $M[G]$ in the language $\langle \in, =, A, B, W \rangle$.*

PROOF. The predicate $W$ defines well ordering of $M$. By Shoenfield [9] $M[G] = \bigcup \{K_G(a) \mid a \in M\}$ and every $K_G(a)$ is a set in $M[G]$. For $b \in M$ let $\Delta(b)$ be defined in $M$ by induction on $\operatorname{rank}(b)$ as follows: $\Delta(b)$ is the smallest $\alpha$ such that $\Delta(a) \leqq \alpha$ for all $a \in \operatorname{rng} b$ and $t \in P_\alpha$ for all $t \in \operatorname{dom} b$. By [9] $\Delta(b) \leqq \alpha \rightarrow K_G(b) = K_{G_\alpha}(b)$ and $M[G] = \bigcup_{\alpha \in \mathrm{Reg}} M[G_\alpha]$. Hence $G_\alpha \in M[G]$ for every $\alpha \in \mathrm{Reg}$ and there is a formula $\varphi_0(x, y)$ such that $M[G] \vDash \varphi_0(x, y)$ iff $y \in 0n$ and $x \in M[G_y]$. There is also a formula $\varphi_1(x, y, z)$ such that $M[G] \vDash \varphi_1(x, y, z)$ iff $M[G] \vDash (\varphi_0(x, y)$ and $x = K_{G_y}(z))$. Notice that for the definition of $K_{G_y}(z)$ we need the axiom of replacement only in $M[G_y]$, which is a model of ZFC. Write now $\varphi_2(x, y)$ for $B(x)$ and $((\forall \alpha \in 0n)(\alpha \geqq \Delta(y)$ $\rightarrow (\varphi_0(y, \alpha)$ and $\varphi_1(y, \alpha, x))$. Then $M[G] \vDash \varphi_2(x, y)$ iff $y = K_G(x)$. Combining $W$ with $\varphi_2$ we get a definable well ordering of $M[G]$. □

Let us fix some such well ordering.

Lemma 4.5. *Let $\varphi(x,y)$ be any formula of the language of $\langle \in, =, A, B, W\rangle$. If for some $a \in M[G]$, $M[G] \models (\forall x \in a)(\exists y)\ (\varphi(x,y))$, then there is $b \in M[G]$ such that $M[G] \models (\forall x \in a)(\exists y \in b)(\varphi(x,y))$.*

Proof. First note that for $\alpha \in \mathrm{Reg}$, $M[G_\alpha]$ is definable in $M[G]$. Let, for $x \in a$, $\psi(x,\alpha)$ be equivalent to the following statement: '$\alpha$ is a least regular cardinal in $M$ such that there is $y \in R_\alpha^{M[G_\alpha]}$ which satisfies $\Phi(x,y)$', where $R_\alpha^{M[G_\alpha]} = \{x \in M[G_\alpha] \mid \mathrm{rank}^{M[G_\alpha]} x < \alpha\}$. Note that for every $x \in \mathrm{dom}\, a$ there is an ordinal $\beta_x$ such that for every $\gamma \geqq \beta_x$, $t \in P_3$, $t \not\Vdash \psi(x, \check{\gamma})$, since otherwise we obtain a proper class of pairwise incompatible conditions which is impossible, since suppose that $\{\langle p_\alpha, U_\alpha\rangle \mid \alpha \in \mathrm{Reg}\} \subseteq P_3$ are pairwise incompatible. Define for such $f(\alpha) = p_\alpha \mid \alpha$. Since $|p_\alpha| < \aleph_0$ we can assume without loss of generality that $f$ is a regressive function. But if $f(\alpha) = f(\beta)$, then $\langle p_\alpha, U_\alpha\rangle$ and $\langle p_\beta, U_\beta\rangle$ are compatible by Lemmas 2.5, 2.7 and 2.8. $\beta = \bigcup\{\beta_x \mid x \in \mathrm{dom}\, a\}$. Let $\alpha$ be the least regular cardinal in $M$ which is $\geqq \beta$. Then $M[G] \models (\forall x \in a)(\exists y \in R_\alpha^{M[G_\alpha]})(\varphi(x,y))$. $R_\alpha^{M[G_\alpha]} \in M[G_\alpha] \subseteq M[G]$ since $M[G_\alpha]$ is the model of ZFC. □

Lemma 4.6. *Let $\Phi(x,y)$ be a formula of the language of $\langle \in, =, B, W\rangle$. If $N_G \models (\forall x \in a)(\exists y)(\Phi(x,y))$ then there is $b \in N_G$ such that $N_G \models (\forall x \in a)$ $(\exists y \in b)(\Phi(x,y))$.*

Proof. For $\alpha \in \mathrm{Reg}$ let $N_{G_\alpha}$ be the symmetric submodel of $M[G_\alpha]$ which is generated by $\{H_a \mid a$ is a finite set of regular cardinals $< \alpha\}$. Then $N_G = \bigcup\{N_{G_\alpha} \mid \alpha \in \mathrm{Reg}\}$. Let in $M[G] \models \psi(x,\alpha)$ iff $\alpha$ is the least regular cardinal in $M$ such that for some $y \in R_\alpha^{N_{G_\alpha}}$ $N_G \models \Phi(x,y)$. By Lemma 4.5 we obtain $\alpha_0$ such that $N_G \models (\forall x \in a)(\exists y \in R_{\alpha_0}^{N_{G_{\alpha_0}}})(\Phi(x,y))$. Note that since $N_{G_{\alpha_0}}$ is the model of ZF, $R_{\alpha_0}^{N_{G_{\alpha_0}}} \in N_{G_{\alpha_0}} \subseteq N_G$. □

Theorem 4.7. (1) *The Axiom of Replacement is true in $M[G]$ for any formula of the language of $\langle \in, =, A, B, W\rangle$.*

(2) *The Axiom of Replacement is true in $N_G$ for any formula of the language of $\langle \in, =, B, W\rangle$.*

Proof. The models $M[G]$, $N_G$ are closed under Gödel's operations. Hence we need only to show that for any formula $\Phi(x,y)$ in the models $M[G]$, $N_G$ the following holds:

$$(\forall a)(\exists b)(((\forall x \in a)(\exists y)(\Phi(x,y)) \leftrightarrow ((\forall x \in a)(\exists y \in b)(\Phi(x,y))))).$$

This follows from 4.5 and 4.6. □

Note that $M[G] \models \mathrm{ZFC} + `\forall \alpha \in On\ |\alpha| \leqq \aleph_0'$, where $\mathrm{ZFC}^-$ is ZFC without the Axiom of Power Set.

## 5. The Axiom of Power Set

The next lemma is based on a modification of theorem 1.36 of Prikry [8].

LEMMA 5.1. *Let $\sigma = \varphi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$ be a formula, $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m \in \mathrm{HS}$ and* $\mathrm{sym}(\langle x_1, \cdots, x_m \rangle) \supseteq H_a$ *for some finite set $a$ of regular cardinals. Then for every forcing condition $\langle r, B \rangle$ such that* $\mathrm{dom}\, r = a$ *there is a stronger condition $\langle s, A \rangle$ such that* $\mathrm{dom}\, s = \mathrm{dom}\, r$, *$s \approx r$ and $\langle s, A \rangle \| \sigma$, where by $\langle s, A \rangle \| \sigma$ (read $\langle s, A \rangle$ decides $\sigma$) we mean that $\langle s, A \rangle \Vdash \sigma$ or $\langle s, A \rangle \Vdash \neg \sigma$).*

PROOF. Let $r_1 \in B$ and $r_1 \approx r$. We shall define for every such $r_1$ a set $F_{r_1} \subseteq B$. If $r_1$ is not extendible then we put $F_{r_1} = \{r_1\}$. If $r_1$ is extendible let

$$S_0^{r_1} = \{s \in B \mid s \supseteq r_1,\ s \mid k_1 = r \mid k_1 \text{ and there exists a } U \subseteq B$$
$$\text{such that } \langle s, U \rangle \in P_3 \text{ and } \langle s, U \rangle \Vdash \sigma\},$$
$$S_1^{r_1} = \{s \in B \mid s \supseteq r_1,\ s \mid k_1 = r_1 \mid k_1 \text{ and there exists a } U \subseteq B$$
$$\text{such that } \langle s, U \rangle \in P_3 \text{ and } \langle s, U \rangle \Vdash \neg \sigma\},$$
$$S_2^{r_1} = \{s \in B \mid s \supseteq r_1,\ s \mid k_1 = r_1 \mid k_1 \text{ and there is no } U \subseteq B \text{ such}$$
$$\text{that } \langle s, U \rangle \in P_3 \text{ and } \langle s, U \rangle \| \sigma\},$$
$$S^{r_1} = S_0^{r_1} \cup S_1^{r_1} \cup S_2^{r_1} = \{s \in B \mid s \supseteq r_1,\ s \mid k_1 = r \mid k_1\}.$$

We shall now consider the double domains ($\mathrm{dom}^2$) of the members $s$ of $S^{r_1}$. We shall denote by $x$ the function on some $k \leqq \omega$ which enumerates the additions to $\mathrm{dom}^2 r_1$ needed to obtain those double domains. Let $a - k_1 = \{\alpha_0, \cdots, \alpha_{n-1}\}$, where $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1}$; since we assumed that $S^{r_1} \supsetneqq \{r_1\}$ we have $n > 0$. We define $x$ by recursion as follows: $x(0) = \langle \alpha(r_1), m(r_1) \rangle$. If $x(i)$ is defined let $x(i) = \langle \alpha_j, m \rangle$ for some $0 \leqq j < n$ and $m < \omega$. Let $\langle \bar{\alpha}, \bar{m} \rangle$ be $\langle \alpha_{j+1}, m \rangle$ if $j < n - 1$ and $\langle \alpha_0, m + 1 \rangle$ if $j = n - 1$. If $\mathrm{cf}' \bar{\alpha} \geqq k_1$ or $\mathrm{cf}' \bar{\alpha} < k_1$ and $\langle \mathrm{cf}' \bar{\alpha}, m \rangle \in \mathrm{dom}^2 r_1$, then set $x(i+1) = \langle \bar{\alpha}, \bar{m} \rangle$; if $\mathrm{cf}' \bar{\alpha} < k_1$ and $\langle \mathrm{cf}' \bar{\alpha}, \bar{m} \rangle \not\in \mathrm{dom}^2 r_1$ then $x(i+1)$ is undefined and $\mathrm{dom}\, x = i + 1$, since by (3) of Definition 2.2 there is no $s \in S^{r_1}$ such that $\langle \bar{\alpha}, \bar{m} \rangle \in \mathrm{dom}^2 s$. Notice that for every $i < \mathrm{dom}\, x$ and $s \in S^{r_1}$ if $x(i) = \langle \alpha(s), m(s) \rangle$ and $\mathrm{cf}' \alpha(s) < \alpha(s)$ then $\langle \mathrm{cf}' \alpha(s), m(s) \rangle \in \mathrm{dom}^2 s$; if

$\mathrm{cf}'\alpha(s) \geqq k_1$ this follows from $s \in P_2$, if $\mathrm{cf}'\alpha(s) < k_1$ and $i > 0$ this follows from the fact that $x(i)$ is defined, and if $\mathrm{cf}'\alpha(s) < k_1$ and $i = 0$ this follows from our assumption that $r_1$ is extendible.

For every $i < \mathrm{dom}\, x$ let $\tilde{F}_i = \{s \in S^{r_1} \mid \mathrm{dom}^2 s - \mathrm{dom}^2 r_1 = x''(i+1)\}$. We define now a set of functions $F_{i,j}$ on $\tilde{F}_i$ for every $i, j$ such that $\mathrm{dom}\, x > j \geqq i$. For $l < 3$ and $s \in \tilde{F}_i$ let $F_{i,i}(s) = l$ if $s \in S_l^{r_1}$ and for $i < j$ let $F_{i,j}(s) = l$ if the set

$$\{\beta \mid s \cup \{\langle \alpha(s), m(s), \beta \rangle\} \in B \text{ and}$$

$$F_{i+1,j}(s \cup \{\langle \alpha(s), m(s), \beta \rangle\}) = l\}$$

is a member of the ultrafilter $\Phi_s$.

We shall now define $\tilde{F}'_i \subseteq \tilde{F}_i$ for $i < \mathrm{dom}\, x$, by recursion. Using the $k_1$-completeness of the ultrafilter $\Phi_{r_1}$ we find a set $\tilde{F}'_0 \subseteq \tilde{F}_0$ which is homogeneous for all functions $F_{0,j}$ ($j < \mathrm{dom}\, x$) (homogeneity means that for all $t_1, t_2 \in \tilde{F}'_0$ and for all $0 \leqq j < \mathrm{dom}\, x$, $F_{0,j}(t_1) = F_{0,j}(t_2)$) such that $\{\beta \mid r_1 \cup \{\langle \alpha(r_1), m(r_1), \beta \rangle\} \in \tilde{F}'_0\} \in \Phi_{r_1}$.

For $0 < i < \mathrm{dom}\, x$ we take $\tilde{F}'_i = \{s \in \tilde{F}_i \mid s^- \in \tilde{F}'_{i-1}$ and for all $i \leqq j < \mathrm{dom}\, x$, $F_{i,j}(s) = F_{i-1,j}(s^-)\}$ where $s^-$ is as in (P10). It follows easily by induction on $i$ that $\tilde{F}'_i$ is homogeneous for all functions $F_{i,j}$ ($i \leqq j < \mathrm{dom}\, x$). The definition of the functions $F_{i,j}$ implies that for every $t \in \tilde{F}'_{i-1}$, $\{\beta \mid t \cup \{\langle \alpha(t), m(t), \beta \rangle\} \in \tilde{F}'_i\} \in \Phi_t$. Set $\tilde{F}_{r_1} = \{r_1\} \cup \bigcup \{\tilde{F}'_i \mid i < \mathrm{dom}\, x\}$. To see that this construction leads us in the right direction we shall show that if $s_1, s_2 \in \tilde{F}_{r_1}$, $\langle s_1, A_1 \rangle, \langle s_2, A_2 \rangle \in P_3$ and $\langle s_1, A_1 \rangle, \langle s_2, A_2 \rangle \geqq \langle r, B \rangle$ then we cannot have $\langle s_1, A_1 \rangle \Vdash \sigma$, $\langle s_2, A_2 \rangle \Vdash \neg\sigma$. For some $i_1, i_2 \leqq \mathrm{dom}\, x$, $\mathrm{dom}^2 s_j = \mathrm{dom}^2 r_1 \cup \{x(k) \mid k < i_j\}$ for $j = 1, 2$. Assume, without loss of generality, that $i_1 \leqq i_2$. We shall see that if $i_1 < i_2$ we can increase $\mathrm{dom}^2 s_1$ single steps till we get $i_1 = i_2$. Since $\langle \alpha(s_1), m(s_1) \rangle = x(i_1)$ and $i_1 < i_2 < \mathrm{dom}\, x$, $s_1$ is extendible. By what we have mentioned $E = \{\beta < \alpha(s_1) \mid s_1 \cup \{\langle \alpha(s_1), m(s_1), \beta \rangle\} \in \tilde{F}'_{i_1}\} \in \Phi_{s_1}$ and since $\langle s_1, A_1 \rangle \in P_3$, $E' = \{\beta < \alpha(s_1) \mid s_1 \cup \{\alpha(s_1), m(s_1), \beta\rangle \in A_1\} \in \Phi_{s_1}$. Let $\beta \in E \cap E'$, let $\bar{s}_1 = s_1 \cup \{\langle \alpha(s_1), m(s_1), \beta \rangle\}$ and let $\bar{A}_1 = \{t \in A_1 \mid t \supseteq s_1\}$. Then clearly $\bar{s}_1 \in \tilde{F}'_{i_1} \subseteq \tilde{F}_{r_1}$ and $\langle \bar{s}_1, \bar{A}_1 \rangle \geqq \langle s_1, A_1 \rangle$, hence $\langle \bar{s}_1, \bar{A}_1 \rangle \Vdash \sigma$ and $\mathrm{dom}^2 \bar{s}_1 = \mathrm{dom}^2 r_1 \cup \{x(k) \mid k < i_1 + 1\}$. This way we keep increasing $i_1$ till we get $i_1 = i_2$. We denote $i_1 - i_2$ by $i$. If $i = 0$ then $s_1 = s_2 = r_1$ hence $\langle s_1, A_1 \rangle$ and $\langle s_2, A_2 \rangle$ are compatible, contradicting $\langle s_1, A_1 \rangle \Vdash \sigma$, $\langle s_2, A_2 \rangle \Vdash \neg\sigma$. If $i > 0$ then $s_1, s_2 \in \tilde{F}'_{i-1}$. Since $\langle s_1, A_1 \rangle \geqq \langle r, B \rangle$ and $\langle s_1, A_1 \rangle \Vdash \sigma$ we have $s_1 \in S_0^{r_1}$, hence $F_{i-1,i-1}(s_1) = 0$, and similarly $F_{i-1,i-1}(s_2) = 1$ contradicting the homogeneity of $\tilde{F}'_{i-1}$ for $F_{i-1,i-1}$.

Let $\tilde{F} = \bigcup \{\tilde{F}_{r_1} \mid r_1 \in B$ and $r_1 \approx r\}$, $\langle r, \tilde{F} \rangle$ satisfies all the requirements of membership in $P_3$ except possibly (P6). This is the reason why we define yet another set $\tilde{F}'$. First we put in $\tilde{F}'$ $r$ as well as every $r' \in B$ such that $r' \approx r$. For $s \in \tilde{F}'$, we put $s \cup \{\langle \alpha(s), m(s), \beta \rangle\}$ in $\tilde{F}'$ iff for every $r_1 \in B$ such that $r_1 \approx r$ and $r_1 \supseteq s \mid k_1$, $r_1 \cup s \cup \{\langle \alpha(s), m(s), \beta \rangle\} \in \tilde{F}_{r_1}$.

To prove that $\tilde{F}' \subseteq \tilde{F}$ let $s \in \tilde{F}'$. If $s \approx r$ then, by the definition of $\tilde{F}'$, we have $s \in B$ and therefore $s \in \tilde{F}_s \subseteq \tilde{F}$. If $s \not\approx r$ and $s = s^- \cup \{\langle \alpha(s^-), m(s^-), \beta \rangle\} \in \tilde{F}'$ then, by the definition of $\tilde{F}'$ $s^- \in \tilde{F}'$ and for every $r_1 \in B$ such that $r_1 \approx r$ and $r_1 \supseteq s^- \mid k_1$, $r_1 \cup s \in \tilde{F}_{r_1} \subseteq \tilde{F}$. We may assume as an induction hypothesis that $s^- \in \tilde{F} \subseteq B$. Taking $r_1 = r \cup s^- \mid k_1$ (which is in $B$ by requirements (P6) and (P7) of Definition 2.3) we get that $s = r_1 \cup s \in \tilde{F}$.

We shall now prove

(1) $\qquad$ if $s \in \tilde{F}', \bar{r} \in B, \bar{r} \approx r$ and $\bar{r} \supseteq s \mid k_1$ then $s \cup \bar{r} \in \tilde{F}'$.

We shall prove (1) by induction on the number $k$ of pairs $\langle \alpha, m \rangle$, with $\alpha \geqq k_1$, in $\operatorname{dom}^2 s - \operatorname{dom}^2 r$. If $k = 0$ then $s \approx r$, and since $\bar{r} \supseteq s \mid k_1$, $\bar{r} \supseteq s$ and we have indeed $s \cup \bar{r} = \bar{r} \in \tilde{F}'$ by the definition of $\tilde{F}'$. If $k > 0$, then $s \not\approx r$ and we have, by the definition of $\tilde{F}'$, $s = s^- \cup \{\langle \alpha(s^-), m(s^-), \beta \rangle\}$, for some $\beta < \alpha(s^-)$, $s^- \in \tilde{F}'$ and for every $r_1 \in B$ such that $r_1 \approx r$ and $r_1 \supseteq s^- \mid k_1$, $s \cup r_1 \in F_{r_1}$. Since $\bar{r} \in B$, $\bar{r} \approx r$ and $\bar{r} \supseteq s \mid k_1 = s^- \mid k_1$ and (1) is assumed to hold for $s^-$ by the induction hypothesis, we have $s^- \cup \bar{r} \in \tilde{F}' \subseteq \tilde{F}$. Since $\bar{r} \approx r$ we have $\alpha(s^- \cup \bar{r}) = \alpha(s^-)$ and $m(s^- \cup \bar{r}) = m(s^-)$ and therefore in order to prove that $s \cup \bar{r} = s^- \cup \bar{r} \cup \{\langle \alpha(s^-), m(s^-), \beta \rangle\} \in \tilde{F}'$ all we have to do, by the definition of $\tilde{F}'$, is to show that for every $r_1 \in B$ such that $r_1 \approx r$ and $r_1 \supseteq \bar{r}$ we have $r_1 \cup s \cup \bar{r} = r_1 \cup s \in F_{r_1}$. But this follows directly from the facts that $s \in \tilde{F}'$ and $s \not\approx r$, by the definition of $\tilde{F}'$.

Now let us prove that $\langle r, \tilde{F}' \rangle \in P_3$. Requirements (P1)–(P5), (P7) and (P10) hold obviously. Requirement (P6) follows easily from (1) above. Now let us prove that (P8) and (P9) also hold. Let $s \in \tilde{F}'$ be such that either $\operatorname{cf}' \alpha(s) \geqq k_1$ or else $\operatorname{cf}' \alpha(s) < k_1$ and $\langle \operatorname{cf}' \alpha(s), m(s) \rangle \in \operatorname{dom}^2 s$. For $r_1 \in B$ such that $r_1 \approx r$, $r_1 \supseteq s \mid k_1$ let $E_{r_1} = \{\beta < \alpha(s) \mid r_1 \cup s \cup \{\langle \alpha(s), m(s), \beta \rangle\} \in \tilde{F}_{r_1}\}$. By (1), $r_1 \cup s \in \tilde{F}' \subseteq \tilde{F}$. Since $r_1 \cup s \in \tilde{F}$ there is an $i < \operatorname{dom} x$ such that $r_1 \cup s \in \tilde{F}'_i$. By our assumption $s$ can be extended on $\langle \alpha(s), m(s) \rangle$ in $B$, hence $\operatorname{dom} x > i + 1$. As we have shown above, $E_{r_1}$ belongs to the ultrafilter $\Phi_{r_1 \cup s}$. By our assumption on $s$ and since $r_1 \approx r$, $r_1 \supseteq s \mid k_1$ we have $\Phi_{r_1 \cup s} = \Phi_s$. $\Phi_s$ is $k_1$-complete and there are at most $|\bigcup(k_1 \cap \operatorname{dom} r)|^{\aleph_0} + 1 < k_1$, $r_1 \in B$ such that $r_1 \approx r$. Hence $E = \bigcap \{E_{r_1} \mid r_1 \in B$ and $r_1 \approx r$ and $r_1 \supseteq s \mid k_1\} \in \Phi_s$. By the definition of $\tilde{F}'$ we have $\{\beta < \alpha(s) \mid s \cup \{\langle \alpha(s), m(s), \beta \rangle\} \in \tilde{F}'\} = E \in \Phi_s$. Hence (P8) and (P9) hold for $\langle r, \tilde{F}' \rangle$.

Finally, let $\langle s_1, A_1 \rangle \geqq \langle r, \tilde{F}' \rangle$ be any condition which decides $\sigma$. By Lemma 3.3 we can assume that $\langle s_1, A_1 \rangle \mid \operatorname{dom} r = \langle s_1, A_1 \rangle$ and hence $s_1 \in \tilde{F}'$, $A_1 \subseteq \tilde{F}'$. Put

$s = s_1 \mid k_1 \cup r$, $A = \{t \in F' \mid t \supseteq s$ and $t \mid k_1 \cup s_1 \in A_1\}$. We shall show that $\langle s, A\rangle \| \sigma$. Assume that $\langle s_1, A_1\rangle \Vdash \sigma$ (if $\langle s_1, A_1\rangle \Vdash \neg\sigma$ we can proceed similarly). Suppose that $\langle s, A\rangle \nVdash \sigma$. Then there is a condition $\langle s_2, V\rangle \geqq \langle s, A\rangle$ which forces $\neg\sigma$. By Lemma 3.3 it is enough to only deal with the case where $s_2 \in A$ and $V \subseteq A$. Let $s_1' = s_1 \cup s_2 \mid k_1$. By the definition of $A$, $s_1' \in A_1 \subseteq F'$. Put $A' = \{q \in A_1 \mid q \supseteq s_1'\}$. Then $\langle s_1', A'\rangle \in P_3$, $\langle s_1', A'\rangle \geqq \langle s_1, A_1\rangle$ and, hence, $\langle s_1', A'\rangle \Vdash \sigma$. We have $s_1', s_2 \in F_{r \cup s_2 \mid k_1}$, $\langle s_1', A'\rangle, \langle s_2, V\rangle \geqq \langle r, \tilde{F}'\rangle$, $\langle s_1', A'\rangle \Vdash \sigma$ and $\langle s_2, V\rangle \Vdash \neg\sigma$, but this contradicts what we proved above about the $F_{r_1}$'s.

Definition 5.2. Let $\langle p, U\rangle, \langle s, V\rangle \in P_3$ and $q \in P_2$ be such that

(1) $\operatorname{dom} q \supseteq \operatorname{dom} p \cup \operatorname{dom} s$,

(2) $q \mid \operatorname{dom} p \in U$,

(3) $q \mid \operatorname{dom} s \in V$.

Then let $\langle s, V\rangle \cup^q \langle p, U\rangle$ be the ordered pair $\langle q, W\rangle$, where

$$W = \{t \in P_2 \mid t \supseteq q, \operatorname{dom} t = \operatorname{dom} q, t \mid \operatorname{dom} p \in U, t \mid \operatorname{dom} s \in V\}.$$

Lemma 5.3. *Such a defined pair $\langle q, W\rangle$ is a forcing condition stronger than both $\langle p, U\rangle$ and $\langle s, V\rangle$.*

Proof. Let $p' = q \mid \operatorname{dom} p$ and $s' = q \mid \operatorname{dom} s$, then by Lemma 2.5 $\langle p', U_{p'}\rangle$, $\langle s', V_{s'}\rangle \in P_3$, where $U_{p'} = \{r \in U \mid r \supseteq p'\}$, $V_{s'} = \{r \in V \mid r \supseteq s'\}$. By Lemma 2.6 $\langle q, U'\rangle$, $\langle q, V'\rangle \in P_3$ where $U' = \{r \mid r \supseteq q$ and $r$ is a $\operatorname{dom} q$-extension of some $t \in U_{p'}\}$ and $V' = \{r \mid r \supseteq q$ and $r$ is a $\operatorname{dom} q$-extension of some $t \in V_{s'}\}$. By Lemma 2.7 $\langle p, U\rangle \cup^q \langle s, V\rangle = \langle q, U' \cap V'\rangle \in P_3$. □

Theorem 5.4. *$N_G \models$ the power set axiom.*

Proof. Let $x \in N_G$. We shall construct a one-one definable in $M[G]$ function from $P^{N_G}(x) = P(x) \cap N_G$ into a set in $M$, let $k$ be the least strongly compact cardinal greater than $\delta = \bigcup\bigcup\{\operatorname{supp} z \mid z \in \operatorname{dom} x\} \cup |\operatorname{dom} x|$ where $\operatorname{supp} z$ is the least finite subset $a$ of Reg such that $\operatorname{sym} z \supseteq H_a$.

Put $P_\alpha = \{\langle p, U\rangle \in P_3 \mid \operatorname{dom} p \subseteq \alpha\}$ for regular cardinal $\alpha$. Then, as can be easily seen, $G_\alpha = G \cap P_\alpha$ is a generic subset of $P_\alpha$.

Let $T_k = \{\langle p, U\rangle \in P_3 \mid \forall q \in U, \forall \alpha \geqq k, \forall n < \omega$ if $\langle \alpha, n\rangle \in \operatorname{dom}^2 q - \operatorname{dom}^2 p$ and $\operatorname{cf}'\alpha < \alpha$ then the ultrafilter $\Phi_{\alpha, q(\operatorname{cf}'\alpha)(n)}$ is $k$-complete$\}$. We shall prove that $T_k$ is dense in $P_3$. First notice that the class

$$D = \{\langle p, U\rangle \in P_3 \mid \text{for all } \alpha \in \operatorname{dom} p, \operatorname{dom} p(\alpha) = \operatorname{dom} p(\operatorname{cf}'\alpha)\}$$

is dense in $P_3$, as we saw in Definition 3.1 where we mentioned that $P^{\pi}$ is dense. Now we shall prove that for every $\langle p, U\rangle \in D$ there is a $U' \subseteq U$ such that $\langle p, U'\rangle \in T_k$. For every $\alpha \in \operatorname{dom} p - k$ such that $\mathrm{cf}'\alpha < \alpha$ let $\lambda_\alpha$ be the least ordinal $\nu < \mathrm{cf}'\alpha$ such that $k^\alpha_\nu \geqq k$ where $k^\alpha_\gamma$ is as in Definition 1.1. Clearly, $\langle p, U'\rangle \in T_k$ iff for all $q \in U'$ if $\alpha(q) \geqq k$ and if $\mathrm{cf}'\alpha(q) < \alpha(q)$ then $q(\mathrm{cf}'\alpha(q))(m(q)) \geqq \lambda_{\alpha(q)}$. Therefore if we set $b = \{\mathrm{cf}'\alpha \mid \alpha \in \operatorname{dom} p - k\}$, and for $\bar{\alpha} \in b$ $c_{\bar{\alpha}} = \max\{\lambda_\alpha \mid \alpha \in \operatorname{dom} p - k \text{ and } \mathrm{cf}'\alpha = \bar{\alpha}\}$, then $\langle p, U'\rangle \in T_k$, if for all $q \in U'$ and $\langle \bar{\alpha}, m\rangle \in \operatorname{dom}^2 q - \operatorname{dom}^2 p$, we have $q(\bar{\alpha})(m) \geqq c_{\bar{\alpha}}$. Therefore, we choose $U' = \{q \in U \mid \text{for all } \bar{\alpha} \in b \text{ and } m < \omega \text{ if } \langle \bar{\alpha}, m\rangle \in \operatorname{dom}^2 q - \operatorname{dom}^2 p, \text{ then } q(\bar{\alpha})(m) \geqq (\bar{\alpha})\}$. It is easily seen that $\langle p, U'\rangle \in P_3$.

Let $y \in P^{N_G}(x)$ and $\mathrm{sym}(\langle x, y\rangle) \supseteq H_{\langle \alpha_1, \cdots, k_1, \alpha_n, \cdots, \alpha_r, k, \cdots, \alpha_m\rangle}$ for some regular cardinals $\alpha_1, \cdots, \alpha_n, \cdots, \alpha_m$ such that $n > 0$, $\alpha_1 < \alpha_2 < \cdots < k_1 < \alpha_n \cdots \alpha_r < k < \cdots < \alpha_m$. Let $r_0$ be the least member of $T_k \cap G$, in the well ordering of $M[G]$ such that $r_0 \Vdash y \subseteq x$ and $\operatorname{dom}((r_0)_1) = \{\alpha_0, \cdots, k_1, \alpha_n, \cdots, \alpha_r, k, \cdots, \alpha_m\}$. Such $r_0$ exists by Lemma 3.3. Let $\lambda$ be the least cardinal greater than $\delta$ and $\alpha_r$.

Consider the following statement:

(*) for every condition $\langle p, U\rangle \in T_k$ such that $\langle p, U\rangle \geqq r_0$ and $\operatorname{dom} p \subseteq \lambda^+ \cup (0n - k)$ there is a stronger condition $\langle p, \tilde{F}\rangle$ such that for every $z \in \operatorname{dom} x$ and for every $s \in P_{\lambda^+}$ if $s \geqq \langle p \mid \lambda^+, \tilde{F} \mid \lambda^+\rangle$ and $\operatorname{supp} z \subseteq \operatorname{dom}((s)_1)$ then

(a) There is a forcing condition $s' \in P_{\lambda^+}$ such that

(1) $s' \geqq s$,

(2) $\operatorname{dom}((s')_1) = \operatorname{dom}((s)_1)$,

(3) $(s')_1 \approx (s)_1$,

(4) for some $q \in P_2$ such that $q \mid \lambda^+ = (s')_1$ and $q \mid \operatorname{dom} p \in \tilde{F}$

$$s' \cup^q \langle p, \tilde{F}\rangle \| z \in y.$$

(b) For every $s' \in P_{\lambda^+}$ which satisfies conditions (1)–(4) above, if $q_1, q_2 \in P_2$ are such that $q_1 \mid \lambda^+ = q_2 \mid \lambda^+ = (s')_1$ and $q_1 \mid \operatorname{dom} p$, $q_2 \mid \operatorname{dom} p \in \tilde{F}$, then

$$\begin{aligned} s' \cup^{q_1} \langle p, \tilde{F}\rangle \Vdash z \in y \quad & \text{iff} \\ s' \cup^{q_2} \langle p, \tilde{F}\rangle \Vdash z \in y \quad & \text{and} \\ s' \cup^{q_1} \langle p, \tilde{F}\rangle \Vdash z \notin y \quad & \text{iff} \\ s' \cup^{q_2} \langle p, \tilde{F}\rangle \Vdash z \notin y. & \end{aligned}$$

Suppose that (*) holds. Let $\langle p(y), \tilde{F}(y)\rangle$ be the least member of $G$ in the well-ordering of $M[G]$ which satisfies (*). The intuitive meaning of (*) is that in order to decide statements about membership in $y$, one has just to get information about $G \cap P_{\lambda^+}$.

We define the map $g_y \in M$ from $P_k$ dom $x$ into the set $3 = \{0, 1, 2\}$ as follows:

$$g_y(s, z) = \begin{cases} 1, & \text{if } s \geqq \langle p(y) \mid k, \tilde{F}(y) \mid k \rangle, \operatorname{dom}(s_1) \supseteq \operatorname{supp} z \text{ and for some } q \in P_2 \text{ such that } q \mid \operatorname{dom}(p(y)) \in \tilde{F}(y) \text{ and } q \mid \lambda^+ = (s)_1 \mid \lambda^+ \text{ we have} \\ & \qquad (s \mid \lambda^+) \cup^q \langle p(y), \tilde{F}(y) \rangle \Vdash z \in y \\ 0, & \text{if } s \geqq \langle p(y) \mid k, \tilde{F}(y) \mid k \rangle, \operatorname{dom}(s)_1 \supseteq \operatorname{supp} z \text{ and for some } q \in P_2 \text{ such that } q \mid \operatorname{dom}(p(y)) \in \tilde{F}(y) \text{ and } q \mid \lambda^+ = (s)_1 \mid \lambda^+ \text{ we have} \\ & \qquad (s \mid \lambda^+) \cup^q \langle p(y), \tilde{F}(y) \rangle \Vdash z \notin y \\ 2, & \text{in all other cases.} \end{cases}$$

Note that part (b) of (*) guarantees that $g_y$ is a function. Put $f(y) = g_y$, where $y$ is the least symmetric name of $y$ in the well ordering of $M[G]$. Now we want to show that $f$ is 1–1 map. Let $y_1, y_2 \in P^{N_G}(x)$ and $y_1 \neq y_2$. Find the least symmetric names $y_1, y_2$ for $y_1, y_2$. There are a condition $s_0 \in G$ and a $z \in \operatorname{dom} x$ such that $s_0$ is stronger than $\langle p(y_1), \tilde{F}(y_2) \rangle$ and $\langle p(y_2), \tilde{F}(y_2) \rangle$, $\operatorname{dom}(s_0)_1 \supseteq \operatorname{supp} z$ and $s_0 \Vdash (z \in y_1 \text{ and } z \notin y_2)$ or $s_0 \Vdash (z \in y_2 \text{ and } z \notin y_1)$. Note that $s_0$ also forces the formula $(y_1 \subseteq x$ and $y_2 \subseteq x)$ since $s_0 \geqq \langle p(y_1), \tilde{F}(y_1) \rangle$, $\langle p(y_2), \tilde{F}(y_2) \rangle$. We can assume, without loss of generality, that $s_0 \Vdash (z \in y_1 \text{ and } z \notin y_2)$. Let $\lambda_i$ be the least cardinal greater than $\delta \cup \bigcup (\operatorname{supp}(\langle x, y_i \rangle) \cap k)$ for $i = 1, 2$. Let $s' \geqq s_0 \mid \lambda_1^+$ be a condition which satisfies requirements (1)–(4) of (*) with respect to $\lambda_1$, $s_0 \mid \lambda_1^+$ and $\langle p(y_1), \tilde{F}(y_1) \rangle$. Since $s' \geqq s_0 \mid \lambda_1^+$ and $\operatorname{dom}((s')_1) = \operatorname{dom}((s_0)_1) \cap \lambda_1^+$ we can choose $q \in (s_0)_2$ such that $q \mid \lambda_1^+ = (s_1)_1$. Notice that $q \mid \operatorname{dom}(p(y_i)) \in \tilde{F}_i(y_i)$ since $s_0 \geqq \langle p(y_i), \tilde{F}(y_i) \rangle$ for $i = 1, 2$. Hence $s' \cup^q \langle p(y_i), \tilde{F}(y_i) \rangle$ is defined. Since $s'$ satisfies (1)–(4) of (*) there is a $q^* \in P_2$ such that $q^* \mid \lambda_1^+ = (s')_1$, $q^* \mid \operatorname{dom} p(y_1) \in \tilde{F}(y_1)$ and $s' \cup^{q^*} \langle p(y_1), \tilde{F}(y_1) \rangle \parallel z \in y_1$. By (b) of (*) we have that also $s' \cup^q \langle p(y_1), \tilde{F}(y_1) \rangle \parallel z \in y_1$. Since $q \in (s_0)_2$ we get, by Lemmas 2.5 and 2.8, that $s' \cup^q \langle p(y_1), \tilde{F}(y_1) \rangle$ is compatible with $s_0$. Therefore, since $s_0 \Vdash z \in y_1$ also $s' \cup^q \langle p(y_1), \tilde{F}(y_1) \rangle \Vdash z \in y_1$. But $s' \mid \lambda_2^+ \cup^q \langle p(y_2), \tilde{F}(y_2) \rangle$ is compatible with $s_0$. Hence $s' \mid \lambda_2^+ \cup^q \langle p(y_2), \tilde{F}(y_2) \rangle \nVdash z \in y_2$. Now by the definition of $g$, $g_{y_1}(s' \mid k, z) = 1$ and $g_{y_2}(s' \mid k, z) \neq 1$.

The map $f^{-1}$ is definable from parameters in $M[G]$ and maps a subset of $3^{(P_k \times \operatorname{dom} x)} \cap M$ on $P^{N_G}(x)$. Hence by Theorem 4.7 (1) $P^{N_G}(x) \in M[G]$. For

$y \in P^{N_G}(x)$ let $f'(y)$ be the least strongly compact cardinal $\alpha$ in $M$ such that $y \in N_{G_\alpha}$. By 4.7 (1) we obtain that $P^{N_G} \subseteq N_{G_{\alpha_0}}$, for some $\alpha_0 \in \mathrm{Reg}$. Then since $N_{G_{\alpha_0}} \models \mathrm{ZF}$, $P^{N_G}(x) \in N_{G_{\alpha_0}} \subseteq N_G$.

Hence the theorem will be established once we prove the statement (*).

First we prove two lemmas. Let $\lambda$ be any regular cardinal $> k_1$ and $k$ be the least strongly compact cardinal greater than $\lambda$. Let $\langle p, U\rangle \in P_3$ be such that $\operatorname{dom} p \cap (\lambda - k_1) \neq \emptyset$ and $\operatorname{dom} p \cap (k - \lambda) = \emptyset$. Let $\langle p, V\rangle \in P_3$, $\langle p, V\rangle \geqq \langle p, U\rangle$ we say that $q \in V$ is $\lambda$*-extendible* in $V$ if there is a $q' \in V$ such that $q' \supsetneqq q$ and $q' \mid \lambda = q \mid \lambda$, otherwise we say that $q$ is $\lambda$*-inextendible* in $V$. (Notice that $q$ is $k_1$-extendible in $V$ iff it is extendible in $V$.) By our assumption that $\operatorname{dom} p \cap (\lambda - k_1) \neq \emptyset$ and by the definition of $P_2$ every member $q'$ of $V$ is included in $\lambda$-inextendible member $q$ of $V$ such that $q \mid \lambda = q' \mid \lambda$. It is also easily seen that if $q$ and $\bar{q}$ are $\lambda$-inextendible members of $V$ and $q \mid \lambda = \bar{q} \mid \lambda$ then $\operatorname{dom}^2 q = \operatorname{dom}^2 \bar{q}$. Let $p \subseteq s \subseteq q' \in V$ and $q' \mid \lambda = s \mid \lambda$, where $s$ is not necessarily in $P_2$, then by what we have just said there is a $\lambda$-inextendible $q \in V$ such that $p \subseteq s \subseteq q$ and $q \mid \lambda = q' \mid \lambda = s \mid \lambda$, and $\operatorname{dom}^2 q$ does not depend on the particular such $q$ which we choose. We define a function $x^s$ on $|q - s|$ by requiring $x^s(0), x^s(1), \cdots$ to be the members of $\operatorname{dom}^2(q - s)$ ordered in the right lexicographic order.

For $q \in V$ and some subset $T \subseteq \{\beta \mid q \cup \{\langle \alpha(q), m(q), \beta\rangle\} \in V\}$, we shall say that $T$ has *measure* 1 if it belongs to the ultrafilter $\Phi_q$.

For $s$ such that $s = s' \cup q'$ where $q' \in V$, $s' \in V \mid \lambda$ and $s' \supseteq q' \mid \lambda$ we say that $s$ *reaches* $V$ if either $s \in V$ or else if for some $m \leqq \operatorname{dom} x^s$ the set $A^m_s = \{\beta_0 \mid \{\beta_1 \mid \{\beta_{m-1} \mid s \cup \{\langle (x^s(i))_1, (x^s(i))_2, \beta_i\rangle \mid i < m\} \in U\}$ has measure $1\} \cdots$ has measure $1\}$ has measure 1. Notice that if $A^m_s$ has measure 1 and $m < m' \leqq \operatorname{dom} x^s$ then also $A^{m'}_s$ has measure 1. This follows from (P8) and (P9).

Lemma 5.5. *If $\lambda$, $k$ and $\langle p, U\rangle$ are as above and $\langle p, U\rangle \in T_k$ then there is a $V \subseteq U$ such that $\langle p, V\rangle \in P_3$ and for every member $q$ of $V$ and $s \in V \mid k$ such that $s \supseteq q \mid k$ $s \cup q$ reaches $V$.*

Proof. First we shall define subset $U'$ of $U$ so that for every $s \in U' \mid \lambda$, $p \cup s$ reaches $U$.

We put $p$ in $U'$. Then if $q' \in U$ is such that $q' \approx p$ we put also $q'$ in $U'$. Suppose now that $q$ is already in $U'$ and $q \mid k \cup p$ reaches $U$. Let $\langle \alpha, m\rangle = \langle \alpha(q), m(q)\rangle$. If $\alpha \geqq k$ then we put also $q \cup \{\langle \alpha, m, \beta\rangle\}$ in $U'$ provided $q \cup \{\langle \alpha, m, \beta\rangle\} \in U$. Let now $\alpha < k$. By our assumptions $\alpha$ must be less than $\lambda$. Set $\bar{F} = \{\nu < \alpha \mid p \cup q \mid k \cup \{\langle \alpha, m, \nu\rangle\}$ reaches $U\}$. Let us suppose that $\bar{F}$ has measure 0. Then the set $\tilde{F}' = \alpha - \tilde{F}$ has measure 1. Put $q_0 = q \mid \lambda \cup p$ and $n = \operatorname{dom} x^{q_0}$. Notice that by the definitions of $x^s$ and 2.3

$$x^{q_0 \cup \{\langle \alpha, m, \nu_1 \rangle\}} = x^{q_0 \cup \{\langle \alpha, m, \nu_2 \rangle\}} \quad \text{and} \quad x^{q_0 \cup \{\langle \alpha, m, \nu \rangle\}} \mid n = x^{q_0}$$

for any $\nu, \nu_1, \nu_2 < \alpha$. We shall now define by induction on $i$ a sequence $\langle q_i \mid i \leqq n \rangle$ such that $\operatorname{dom}^2(q_i) = \operatorname{dom}^2(q_0) \cup x^{q_0}(i)$, $q_i$ reaches $U$ and for all $\nu \in \tilde{F}'$, $q_i \cup \{\langle \alpha, m, \nu \rangle\}$ does not reach $U$. For $i = 0$, $q_0$ is defined and is as required. For $i < n$ by the induction hypothesis $q_i$ reaches $U$ hence we have $\{\beta \mid q_i \cup \{\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2, \beta \rangle\}$ reaches $U\}$ has measure 1, while for each $\nu \in \tilde{F}'$, $q_i \cup \{\langle \alpha, m, \nu \rangle\}$ does not reach $U$ and therefore $\{\beta \mid q_i \cup \{\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2, \beta \rangle, \langle \alpha, m, \nu \rangle\}$ reaches $U\}$ has measure 0. Since $|\tilde{F}'| \leqq \alpha < k$ and the ultrafilter corresponding to $\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2 \rangle$ is $k$-complete (as $(x^{q_0}(i))_1 \geqq k$ and $\langle p, U \rangle \in T_k$) the set $T = \{\beta \mid q_i \cup \{\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2, \beta \rangle\}$ reaches $U\} - \bigcup_{\nu \in \tilde{F}'} \{\beta \mid q_i \cup \{\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2, \beta \rangle, \langle \alpha, m, \nu \rangle\}$ reaches $U\}$ has measure 1. Let $\gamma \in T$ and set $q_{i+1} = q_i \cup \{\langle (x^{q_0}(i))_1, (x^{q_0}(i))_2, \gamma \rangle\}$; $q_{i+1}$ is as required. We have, therefore, that $q_n$ reaches $U$, and since $\operatorname{dom}^2 q_n = \operatorname{dom}^2 q_0 \cup \operatorname{rng} x^{q_0}$ we get, by the definition of $x^s$, that $q_n$ is a $\lambda$-inextendible member of $U$. Since $q \mid \lambda = q_0 \mid \lambda = q_n \mid \lambda$ and $q$ is $\lambda$-inextendible we have $\operatorname{dom}^2 q = \operatorname{dom}^2 q_n$. Since $\langle \alpha(q), m(q) \rangle = \langle \alpha, m \rangle \not\in \operatorname{dom}^2 q_n$ we get $\langle \alpha(q_n), m(q_n) \rangle = \langle \alpha, m \rangle$. By (P8) and (P9) we have that $\{\nu < \alpha \mid q_n \cup \{\langle \alpha, m, \nu \rangle\} \in U\}$ has measure 1, hence $\{\nu < \alpha \mid q_n \cup \{\langle \alpha, m, \nu \rangle\}$ reaches $U\}$ has measure 1, while for $\nu \in F'$ we have that $q_n \cup \{\langle \alpha, m, \nu \rangle\}$ does not reach $U$, which shows that $\tilde{F}'$ has measure 0 and $\tilde{F}$ has measure 1. We put now in $U'$ all $q \cup \{\langle \alpha, m, \nu \rangle\}$ such that $q \cup \{\langle \alpha, m, \nu \rangle\} \in U$ and $\nu \in \tilde{F}$ (i.e., $p \cup q \mid \lambda \cup \{\langle \alpha, m, \nu \rangle\}$ reaches $U$). Thus we have for this $q \cup \{\langle \alpha, m, \nu \rangle\}$, $p \cup (q \cup \{\langle \alpha, m, \nu \rangle\}) \mid \lambda = p \cup q \mid \lambda \cup \{\langle \alpha, m, \nu \rangle\}$ reaches $U$ since $\nu \in \tilde{F}$.

We shall now see that if $q \in U$ and $q \mid \lambda \in U' \mid \lambda$ then $q \in U'$. We shall construct an ascending sequence $\langle q_i \mid i \leqq l \rangle$, where $q_0 = p \cup q \mid k_1$, $l = |q - q_0|$, $q_i \subseteq q$, $|q_i - q_0| = i$ (hence $q_l = q$) and $q_i \in U'$ for $i \leqq l$. By the definition of $U'$ we have $q_0 \in U'$ hence $q_0$ is as required. For $i < l$ let $\langle \alpha, m \rangle = \langle \alpha(q_i), m(q_i) \rangle$ then for some $\nu < \alpha$, $\langle \alpha, m, \nu \rangle \in q - q_i$ and we set $q_{i+1} = q_i \cup \{\langle \alpha, m, \nu \rangle\}$, obviously $q_{i+1} \in U$. If $\alpha \geqq \lambda$ then by the definition of $U'$, $q_{i+1} \in U'$. Now let us deal with the case where $\alpha < \lambda$. Since $q \mid \lambda \in U' \mid \lambda$ there is a $q' \in U'$ such that $q' \mid \lambda = q \mid \lambda$. Since $\alpha < \lambda$, $\langle \alpha, m, \nu \rangle \in q \mid \lambda = q' \mid \lambda$. Our definition of $U'$ can be viewed as a construction of its members; let $p'$ be that member of $U'$ which occurs in the construction of $q'$ exactly by the addition of $\langle \alpha, m, \nu \rangle$. Since $q' \mid \lambda = q \mid \lambda$ and $q_{i+1} \subseteq q$ we have $p' \mid \lambda = q_{i+1} \mid \lambda$. By the property of $U'$, since $p' \in U'$, $p \cup p' \mid \lambda$ reaches $U$, hence $p \cup q_{i+1} \mid \lambda$ reaches $U$. By our construction of $U'$ since $q_i \in U'$, $q_{i+1} \in U$ and $\alpha < \lambda$ we get $q_{i+1} \in U'$.

As a consequence, if $q \in U'$ then $p \cup q \mid \lambda$ reaches $U'$, since any member of $U$ obtained from $p \cup q \mid \lambda$ by adding triples $\langle \alpha', m', \beta' \rangle$ with $\alpha' > \lambda$, is also a member of $U'$.

We shall show now that $\langle p, U' \rangle \in P_3$. As it is easily seen, $\langle p, U \rangle$ satisfies the conditions (P1)–(P5), (P7)–(P10) show that also (P6) holds. Let $q_1, q_2 \in U'$, $q_2 \approx p$ and $q_1 \cup q_2 \in P_2$. Construct an ascending sequence $\langle v_i \mid i \leqq l \rangle$ where $v_0 = q_1 \mid k_1 \cup q_2$, $l = |q_1 \cup q_2 - v_0|$, $v_i \subseteq q_1 \cup q_2$, $|v_i - v_0| = i$ (hence $v_l = q_1 \cup q_2$) and $v_i \in U'$ for $i \leqq l$. By (P6) and (P7) for $\langle p, U \rangle$ we have $v_0 \in U$ and, hence by the definition of $U'$, $v_0$ is also in $U'$. For $i < l$ let $\langle \alpha, m \rangle = \langle \alpha(v_i), m(v_i) \rangle$ then for some $\nu < \alpha$, $\langle \alpha, m, \nu \rangle$ is in $q_1 \cup q_2 - v_i$. Set $v_{i+1} = v_i \cup \{\langle \alpha, m, \nu \rangle\}$, obviously $v_{i+1} \in U$. If $\alpha \geqq \lambda$ then by the definition of $U'$, $v_{i+1} \in U'$. If $\alpha < \lambda$ then since $\langle \alpha, m, \nu \rangle \in q_1$ which is in $U'$ we have $p \cup ((v_{i+1} \cap q_1) \mid \lambda)$ reaches $U$. Hence by (P6) for $U$, $p \cup v_{i+1} \mid \lambda$ also reaches $U$ and by the definition of $U'$ then $v_{i+1} = v_i \cup \{\langle \alpha, m, \nu \rangle\}$ is in $U'$.

We shall now construct the main set $V \subseteq U'$. Whenever we put a $q \in U'$ into $V$ we shall also verify that for every $s \in U' \mid \lambda$ such that $s \supseteq q \mid \lambda$, $q \cup s$ reaches $U'$. First we put in $V$, $p$ as well as every $q \in U'$ such that $q \approx p$. For such a $q$, $q = p \cup q \mid k_1 = p \cup q \mid \lambda \subseteq p \cup s$ (for any $s \in U' \mid \lambda$ such that $s \supseteq q \mid \lambda$) hence $q \cup s = p \cup s$ reaches $U'$ by the property of $U'$, thus $q \cup s$ reaches $U'$. Assume now that $q \in U'$ is already in $V$, that for every $s \in U' \mid \lambda$ such that $s \supseteq q \mid \lambda$, $q \cup s$ reaches $U'$ and that $q$ is extendible. Let $\langle \alpha, m \rangle = \langle \alpha(q), m(q) \rangle$. If $\alpha \geqq \lambda$ then, by our assumption, $\alpha \geqq k$. Since $\alpha > \lambda$, $q \cup s$ is not a $\lambda$-inextendible member of $U'$ hence $\operatorname{dom} x^{q \cup s} > 0$ and $x^{q \cup s}(0) = \langle \alpha, m \rangle$. Since $q \cup s$ reaches $U'$, $B_s = \{\beta < \alpha \mid q \cup s \cup \{\langle \alpha, m, \beta \rangle\}$ reaches $U'\}$ has measure 1. Since there are less than $\lambda$ functions $s$ as above and the ultrafilter corresponding to $\langle \alpha, m \rangle$ in $q \cup s$ is $k$-complete (since $\alpha \geqq k$ and $\langle p, U \rangle \in T_k$) we have $B = \{\beta < \alpha \mid$ for all $s$ as above $q \cup s \cup \{\langle \alpha, m, \beta \rangle\}$ reaches $U'\}$ has measure 1. For each $\beta \in B$ we put $q \cup \{\langle \alpha, m, \beta \rangle\}$ in $V$. The required property holds obviously. If $\alpha < \lambda$ then let $q \cup \{\langle \alpha, m, \beta \rangle\}$ be in $V$ if it is in $U'$. Then if $s \in U' \mid \lambda$ is such that $s \supseteq (q \cup \{\langle \alpha, m, \beta \rangle\}) \mid \lambda = q \mid \lambda \cup \{\langle \alpha, m, \beta \rangle\}$, by the assumption on $q$, $s \cup q$ reaches $U'$ and $s \cup q = s \cup q \cup \{\langle \alpha, m, \nu \rangle\}$, hence $s \cup (q \cup \{\langle \alpha, m, \nu \rangle\})$ reaches $U'$.

We shall show now that $\langle p, V \rangle$ satisfies the condition (P6), as it is easily seen that it satisfies all other conditions for being in $P_3$. Let $q_1, q_2 \in V$, $q_2 \approx p$ and $q_1 \cup q_2 \in P_2$. We shall construct an ascending sequence $\langle v_i \mid i \leqq l \rangle$ where $v_0 = q_1 \mid k_1 \cup q_2$, $l = |q_1 \cup q_2 - v_0|$, $v_i \subseteq q_1 \cup q_2$. $|v_i - v_0| = i$ and $v_i \in V$ for $i \leqq l$. Notice that $q_1 \cup q_2 \in U'$ since $\langle p, U' \rangle \in P_3$. By (P6) and (P7) for $\langle p, U' \rangle$ we have $v_0 \in U'$ and hence by the definition of $V$, $v_0 \in V$. For $i < l$ let $\langle \alpha, m \rangle = \langle \alpha(v_i), m(v_i) \rangle$ then for some $\nu < \alpha$, $\langle \alpha, m, \nu \rangle$ is in $q_1 \cup q_2 - v_i$. Set $v_{i+1} =$

$v_i \cup \{\langle \alpha, m, \nu\rangle\}$, obviously $v_{i+1} \in U'$. If $\alpha < \lambda$ then by the definition of $V$, $v_{i+1} \in V$. If $\alpha \geqq \lambda$ then for every $s \in U' \mid k$ such that $s \supseteq (q_1 \cap v_i) \mid \lambda$, $s \cup (v_i \cap q_1)$ reaches $U'$. Let now some $s \in U' \mid k'$ is such that $s \supseteq v_{i+1} \mid \lambda$. Then $s \cup v_{i+1} = s \cup (v_i \cap q_1)$ reaches $U'$. Hence by the definition of $V$, $v_{i+1}$ is in $V$.

Now we shall show that for every $q \in V$ and $s \in V \mid k$ such that $s \supseteq q \mid k$, $s \cup q$ reaches $V$. Suppose not. Then we shall find $q \in V$ and $s \in V \mid k$ such that $s \supseteq q \mid k$, $s \cup q$ does not reach $V$ and the pair $\langle |s \mid k_1 - q \mid k_1|, |\operatorname{dom} x^{s \cup q}|\rangle$ is the least for such $s$, $q$ in the lexicographic well ordering of $\omega \times \omega$, where $x^{s\cup q}$ is used for $U'$ and it is defined since $s \cup q$ reaches $U'$ and hence there is $q' \in U'$, $q' \supseteq s \cup q$. Notice that if $s \mid k_1 \neq q \mid k_1$ then since $\langle p, V\rangle \in P_3$, $q' = q \cup s \mid k_1 \in V$ and $s \cup q' = s \cup q$ does not reach $V$. So $s \mid k_1 = q \mid k_1$. $|\operatorname{dom} x^{s\cup q}| > 0$ since otherwise by the definition of $V$, $s \cup q$ is in $V$ (since $s \mid k_1 = q \mid k_1$, $\operatorname{dom}(s - q) \subseteq \lambda$ and $q \in P_2$) but a member of $V$ always reaches $V$. Hence since $s \cup q$ reaches $U'$, $q$ is extendible. Let $\langle \alpha, m\rangle = \langle \alpha(q), m(q)\rangle$. If $\alpha < k$, then since $s \cup q$ reaches $U'$ there is a $q' \in U'$ such that $q' \supseteq s \cup q$ and hence $q' \supseteq q'' = q \cup \{\langle \alpha, m, s(\alpha)(m)\rangle\} \in U'$. By the definition of $V$, $q'' \in V$. But $q'' \cup s = q \cup s$ which does not reach $V$. Hence $\alpha$ must be greater than or equal to $k$. By (P8) and (P9) the set $B = \{\beta \mid q_\beta = q \cup \{\langle \alpha, m, \beta\rangle\} \in V\}$ has measure 1. But for every $\beta$ in $B$, $q_\beta \mid k = q \mid k \subseteq s$ and hence by our assumption on $s$ and $q$, $s \cup q_\beta$ reaches $V$. So $\{\beta \mid s \cup q \cup \{\langle \alpha, m, \beta\rangle\}$ reaches $V\}$ has measure 1. But then $s \cup q$ reaches $V$. Contradiction. $\square$

Lemma 5.6. *Let $\lambda^+$, $k$ be as in the proof of Theorem 5.4, let $\sigma$ denote $z \in y$ and let $\langle p, U\rangle$ be a forcing condition stronger than $r_0$ as in the conclusion of Lemma 5.5. Then there is a condition $\langle p, U_z\rangle \geqq \langle p, U\rangle$ such that the conditions* (1) *and* (2) *below hold*:

(1) *for every $s \in P_{\lambda^+}$ such that $s \geqq \langle p, U_z\rangle \mid \lambda^+$ and $\operatorname{dom}(s)_1 \subseteq \operatorname{dom} p \cup \operatorname{supp} z$, if $q_1, q_2 \in P_2$ are such that $\operatorname{dom}^2 q_1 = \operatorname{dom}^2 q_2$, $q_1 \mid \operatorname{dom} p$, $q_2 \mid \operatorname{dom} p \in U_z$ and $q_1 \mid \lambda^+ = q_2 \mid \lambda^+ = (s)_1$, then there is a set $V_1$ such that $V_1 \mid \operatorname{dom} p \subseteq U_z$ and $\langle q_1, V_1\rangle \parallel \sigma$ iff for some $V_2$ such that $V_2 \mid \operatorname{dom} p \subseteq U_z$, $\langle q_2, V_2\rangle \parallel \sigma$, and $\langle q_1, V_1\rangle \Vdash \sigma$ iff $\langle q_2, V_2\rangle \Vdash \sigma$*;

(2) *if some forcing condition $\langle q_1, V_1\rangle \geqq \langle p, U_z\rangle$ forces $\sigma$ (or $\neg\sigma$), then $\langle q_1, V_1\rangle \mid \lambda^+ \cup^{q_1} \langle p, U_z\rangle$ forces $\sigma$ (or $\neg\sigma$), and if $q' \in P_2$ is such that $\operatorname{dom}^2 q' = \operatorname{dom}^2 q_1$, $q' \mid \lambda^+ = q_1 \mid \lambda^+$ and $q' \mid \operatorname{dom} p \in U_z$ then $\langle q_1, V_1\rangle \mid \lambda^+ \cup^{q'} \langle p, U_z\rangle \Vdash \sigma$ (or $\neg\sigma$).*

Proof. (1) Let $s \in P_{\lambda^+}$, $s \geqq \langle p, U\rangle \mid \lambda^+$ and $\operatorname{dom}(s)_1 \subseteq \operatorname{dom} p \cup \operatorname{supp} z$. For some $q \in P_2$ such that $\operatorname{dom} q = \operatorname{dom} s \cup \operatorname{dom} p$, $q \mid \operatorname{dom} p \in U$, $q \mid \lambda^+ = (s)_1$ and $q$

is $\lambda^+$-inextendible. Let $x^s$ be the function which enumerates $\operatorname{dom}^2 q - \operatorname{dom}^2 p - k \times \omega$ in the right lexicographic order. Set $n = \operatorname{dom} x^s$. Denote by ${}^j\sigma$, for $j \in 2$, the statement $\sigma$ if $j = 0$ and $\neg\sigma$ if $j = 1$.

The arguments below are similar to the arguments in the proof of Lemma 5.1. Notice that if $\bar q \in U$ is such that $\bar q \mid \lambda^+ \subseteq (s)_1$, then there is a $q' \in P_2$ such that $q' \supseteq \bar q$, $q' \mid \operatorname{dom} p \in U$, $\operatorname{dom} q' = \operatorname{dom} p \cup \operatorname{dom}(s)_1$ and $q' \mid \lambda^+ = (s)_1$. Since by Lemma 5.5, $\bar q \cup (s)_1 \mid \operatorname{dom} p$ reaches $U$, hence there is $q'' \in U$, $q'' \supseteq \bar q \cup (s)_1 \mid \operatorname{dom} p$ and $q'' \mid \lambda^+ = (s)_1 \mid \operatorname{dom} p$. Now by Lemma 2.7 we get $q' \supseteq q''$ as required. For every $i < n$ put $\tilde F_i = \{\bar q \in U \mid \operatorname{dom}^2 \bar q - (\operatorname{dom}^2 p \cup k \times \omega) = x^{s''}(i+1)$ and $\bar q \mid \lambda^+ \subseteq (s)_1\}$. Define the functions $F_i$ on $\tilde F_i$ (for $i < n$) as follows. $F_i(\bar q) = l$ if $\{\beta_{i+1} \mid \{\beta_{i+2} \mid \cdots \{\beta_{n-1} \mid$ there is $V$ such that $\langle (s)_1 \cup \bar q \cup \bigcup \{\langle (x(j))_1, (x(j))_2, \beta_j \rangle \mid i < j < n\}, V \rangle$ is a forcing condition stronger than both $\langle p, U \rangle$ and $s$ which forces ${}^l\sigma\}$ has measure $1\} \cdots \}$ has measure $1\}$ has measure 1 where $l < 2$ and $\bar q \in \tilde F_i$. $F_i(\bar q) = 2$ if $\{\beta_{i+1} \mid \cdots \{\beta_{n-1} \mid$ there is no $V$ such that $\langle (s)_1 \cup \bar q \cup \bigcup \{\langle (x(j))_1, (x(j))_2, \beta_j \rangle \mid i < j < n\}, V \rangle$ is a forcing condition stronger than both $\langle p, U \rangle$ and $s$ which forces either $\sigma$ or $\neg\sigma\}$ has measure $1\} \cdots \}$ has measure $1\}$ has measure 1.

Similarly as in Lemma 5.1 we shall define $\tilde F'_i \subseteq \tilde F_i$ for $i < n$. Using $k_1$-completeness of the ultrafilters $\Phi_\alpha$ and $\Phi_{\alpha,\nu}$ for $\alpha \geqq k_1$ and Lemma 5.5 find a subset $\tilde F'_0$ of $\tilde F_0$ which has measure 1 and is homogeneous for the function $F_0$ (i.e. for all $\bar q_1, \bar q_2 \in \tilde F'_0$, $F_0(\bar q_1) = F_0(\bar q_2)$). For $0 < i < n$ we put $\tilde F'_i = \{\bar q \in F_i \mid$ there is $\bar q' \in \tilde F'_{i-1}$ such that $\bar q \supseteq \bar q'$ and $F_i(\bar q) = F_{i-1}(\bar q')\}$. It is easily seen that $\tilde F'_i$ is homogeneous for the function $F_i$ $(i < n)$. The definition of the functions $F_i$ and Lemma 5.5 imply that for every $\bar q \in \tilde F'_{i-1}$ such that $\alpha(\bar q) \geqq k$, $\{\beta \mid \bar q \cup \{\langle \alpha(\bar q), m(\bar q), \beta \rangle\} \in F'_i\} \in \Phi_{\bar q}$.

Set $\tilde F_s = \bigcup \{\tilde F'_l \mid l < n\}$. Now we are ready to define the main set $U_z \subseteq U$. Put in $U_z$ $p$ as well as every $p' \in U$ such that $p' \approx p$. Let now $q \in U_z$ and a pair $\langle \alpha, m \rangle = \langle \alpha(q), m(q) \rangle$. If $\alpha \geqq k$ then put $q \cup \{\langle \alpha, m, \nu \rangle\}$ in $U_z$ if for every $s \in P_{\lambda^+}$ such that $\operatorname{dom}(s)_1 \subseteq \operatorname{dom} p \cup \operatorname{supp} z$, $s \geqq \langle p, U \rangle \mid \lambda^+$ and $(s)_1 \supseteq q \mid \lambda^+$, $q \cup \{\langle \alpha, m, \nu \rangle\} \in \tilde F_s$. Notice, that since there are less than $(2^\lambda)^+ < k$ $s$ as above and the ultrafilter corresponding to $\langle \alpha, m \rangle$ is $k$-complete (since $\alpha \geqq k$ and $\langle p, U \rangle \in T_k$) we have that $\{\nu \mid q \cup \{\langle \alpha, m, \nu \rangle\} \in U_z\}$ has measure 1. In the case $\alpha < k$ we put $q \cup \{\langle \alpha, m, \nu \rangle\} \in U_z$ if $q \cup \{\langle \alpha, m, \nu \rangle\}$ is in $U$.

Verify that the thus defined pair $\langle p, U_z \rangle$ is a forcing condition. The requirements (P1)–(P5), (P7) and (P10) obviously hold. By the conclusion above (P8) and (P9) also hold. We shall prove (P6). Let $q_1, q_2 \in U_z$, $q_2 \approx p$ and $q_1 \cup q_2 \in P_2$. We shall construct an ascending sequence $\langle \bar q_i \mid i \leqq l \rangle$, where $\bar q_0 = q_2 \cup q_1 \mid k_1$, $l = |q_1 \cup q_2 - \bar q_0|$, $\bar q_i \subseteq q_1 \cup q_2$, $|\bar q_i - \bar q_0| = i$ (hence $\bar q_l = q_1 \cup q_2$) and $\bar q_i \in U_z$ for

$i \leqq l$. By the definition of $U_z$ we have $\bar{q}_0 \in U_z$. For $i < l$ let $\langle \alpha, m \rangle = \langle \alpha(\bar{q}_i), m(\bar{q}_i) \rangle$, then for some $\nu < \alpha$, $\langle \alpha, m, \nu \rangle \in q_1 \cup q_2 - \bar{q}_i$ and we set $\bar{q}_{i+1} = \bar{q}_i \cup \{\langle \alpha, m, \nu \rangle\}$, obviously $q_{i+1} \in U$. If $\alpha < k$ then by the definition of $U_z$, $\bar{q}_{i+1} \in U_z$. Now let us deal with the case where $\alpha \geqq k$. By the definition of $U_z$, since $q_1 \in U_z$, for every $s \in P_{\lambda^+}$ such that $\mathrm{dom}(s)_1 \subseteq \mathrm{dom}\, p \cup \mathrm{supp}\, z$, $s \geqq \langle p, U \rangle \mid \lambda^+$ and $(s)_1 \supseteq q_1 \cap \bar{q}_i$, $q_1 \cap \bar{q}_i \cup \{\langle \alpha, m, \nu \rangle\} \in \tilde{F}_s$. If for such $s$, $(s)_1 \supseteq \bar{q}_i$, then, by the definitions of $F_i$ and $\tilde{F}'_i$, for $i < n$, and since $q_1 \cap \bar{q}_i \approx \bar{q}_i$, $\bar{q}_{i+1} = \bar{q}_i \cup \{\langle \alpha, m, \nu \rangle\} \in \tilde{F}_s$. Hence $\bar{q}_{i+1} \in U_z$.

Let now $q_1, q_2 \in P_2$ be such that $\mathrm{dom}^2 q_1 = \mathrm{dom}^2 q_2$, $q_1 \mid \mathrm{dom}\, p$, $q_2 \mid \mathrm{dom}\, p \in U_z$ and $q_1 \mid \lambda^+ = q_2 \mid \lambda^+ = (s)_1$ for some $s \in P_{\lambda^+}$ such that $s \geqq \langle p, U_z \rangle \mid \lambda^+$ and $\mathrm{dom}(s)_1 \subseteq \mathrm{dom}\, p \cup \mathrm{supp}\, z$. Then $q_1 \mid \mathrm{dom}\, p$, $q_2 \mid \mathrm{dom}\, p \in \tilde{F}_s$. The homogeneity of $\tilde{F}_s$ and Lemma 3.3 imply that for some $V_1$, $V_1 \mid \mathrm{dom}\, p \subset U_z \langle q_1, V_1 \rangle \Vdash \sigma$ (or $\sigma$) iff for some $V_2$, $V_2 \mid \mathrm{dom}\, p \subseteq U_z \langle q_2, V_2 \rangle \Vdash \sigma$ (or $\sigma$).

(2) Let some condition $\langle q_1, V_1 \rangle \geqq \langle p, U_z \rangle$ force $\sigma$ (the case with $\neg \sigma$ is similar). We can assume by Lemma 3.3 that $\mathrm{dom}\, q_1 = \mathrm{dom}\, p \cup \mathrm{supp}\, z$. Set $s = \langle q_1, V_1 \rangle \mid \lambda^+$. If $t = s \cup^{q_1} \langle p, U_z \rangle \nVdash \sigma$ then some condition $\langle q_2, V_2 \rangle$ stronger than $t$ forces $\neg \sigma$ and again by Lemma 3.3, we can assume that $\mathrm{dom}\, q_2 = \mathrm{dom}\, p \cup \mathrm{supp}\, z = \mathrm{dom}\, q_1$. Therefore $q_2 \mid \lambda^+ \in (s)_2$ and hence there is $q_3 \in V_1$ such that $q_3 \mid \lambda^+ = q_2 \mid \lambda^+$ and $\mathrm{dom}^2 q_3 = \mathrm{dom}^2 q_2$. Then by (1) there is a set $V_3$, $V_3 \mid \mathrm{dom}\, p \subseteq U_z$, $\langle q_3, V_3 \rangle \Vdash \sigma$. But it is impossible since the conditions $\langle q_3, V_3 \rangle$ and $\langle q_1, V_1 \rangle$ are compatible. Hence $t = s \cup^{q_1} \langle p, U_z \rangle \Vdash \sigma$.

Let $q' \in P_2$ be such that $\mathrm{dom}^2 q' = \mathrm{dom}^2 q_1$, $q' \mid \lambda^+ = q_1 \mid \lambda^+ = (s)_1$ and $q' \mid \mathrm{dom}\, p \in U_z$. If $t' = s \cup^{q'} \langle p, U_z \rangle \nVdash \sigma$ then let $\langle q'', V'' \rangle$ be any stronger condition which forces $\neg \sigma$ and $\mathrm{dom}\, q'' = \mathrm{dom}\, q_1$ (by Lemma 3.3, as above). Since $q'' \mid \lambda^+ \in (s)_2$, there is a $\tilde{q} \in V_1$ such that $\tilde{q} \mid \lambda^+ = q'' \mid \lambda^+$ and $\mathrm{dom}^2 \tilde{q} = \mathrm{dom}^2 q''$. Then by (1) there is a set $\tilde{V}$ such that $\tilde{V} \mid \mathrm{dom}\, p \subseteq U_z$ and $\langle \tilde{q}, \tilde{V} \rangle \Vdash \neg \sigma$. But it is impossible since $\langle \tilde{q}, \tilde{V} \rangle$ and $\langle q_1, V_1 \rangle$ are compatible. □

Proof of (*). Let $\langle p, U \rangle \in T_k$ be a condition stronger than $r_0$ which satisfies the conclusion of Lemma 5.5. For every $z \in \mathrm{dom}\, x$ find by Lemma 5.6 the condition $\langle p, U_z \rangle \geqq \langle p, U \rangle$. Put $V = \bigcap \{U_z \mid z \in \mathrm{dom}\, x\}$. We shall prove that $\langle p, V \rangle$ is a forcing condition. By the definitions of $U_z$ ($z \in \mathrm{dom}\, x$), $p \in V$ and if $q \in U$ and $q \approx p$ then $q$ is also in $V$. Let $q \in V$ and $\langle \alpha, m \rangle = \langle \alpha(q), m(q) \rangle$. If $\alpha < k$ then, again, by the definitions of $U_z$, $q \cup \{\langle \alpha, m, \nu \rangle\} \in V$ provided $q \cup \{\langle \alpha, m, \nu \rangle\} \in U$. If $\alpha \geqq k$ then

$$\{\nu \mid q \cup \{\langle \alpha, m, \nu \rangle\} \in V\} = \bigcap_{z \in \mathrm{dom}\, x} \{\nu \mid q \cup \{\langle \alpha, m, \nu \rangle\} \in U_z\}$$

and it is the intersection of less than $\lambda^+$ sets of measure 1, hence the set of measure 1, since each $\langle p, U_z\rangle \in T_k$. Now we shall show that $\langle p, V\rangle$ satisfies (*). Let $z \in \operatorname{dom} x$, $s \in P_{\lambda^+}$, $s \geqq \langle p, V\rangle \mid \lambda^+$ and $\operatorname{supp} z \subseteq \operatorname{dom}((s)_1)$. Note that $s/\operatorname{dom}(p) \in U_z$. Let $q_1 \in P_2$ be such that $q_1 \mid \lambda^+ = (s)_1$, $q_1 \mid \operatorname{dom} p \in V$ and $\operatorname{dom} q_1 = \operatorname{dom} p \cup \operatorname{dom}((s)_1)$. Find by Lemma 5.1 a condition $t \geqq s \cup^{q_1} \langle p, V\rangle$ which decides $z \in y$, and which satisfies the conclusion of that Lemma. Set $s' = t \mid \lambda^+$, $q = (t)_1$. Then $s' \geqq s$, $\operatorname{dom}((s')_1) = \operatorname{dom}((s)_1)$, by Lemma 5.1 $(s')_1 \approx (s)_1$ and by Lemma 5.6 $s' \cup^{q} \langle p, V\rangle \| z \in y$.

Let now some $s' \in P_{\lambda^+}$ satisfies requirements (1)–(4) of (*). Let $q_1, q_2 \in P_2$ be such that $q_1 \mid \lambda^+ = q_2 \mid \lambda^+ = (s')_1$ and $q_1 \mid \operatorname{dom} p$, $q_2 \mid \operatorname{dom} p \in V$. Suppose that $t_1 = s' \cup^{q_1} \langle p, V\rangle \Vdash z \in y$ and $t_2 = s' \cup^{q_2} \langle p, V\rangle \nVdash z \in y$. Find $t_3 \geqq t_2$, $t_3 \Vdash z \notin y$. We may assume that $\operatorname{dom}((t_3)_1) = \operatorname{dom} p \cup \operatorname{supp} z = \operatorname{dom}((t_1)_1)$. Extend $q_1$, $(t_3)_1$ until $q_1', q_2'$ so that $q_1' \in (t_1)_2$, $q_2' \in (t_3)_2$, $q_1' \mid \lambda^+ = q_2' \mid \lambda^+$ and $\operatorname{dom}^2 q_1' = \operatorname{dom}^2 q_2'$. Then $\langle q_1', \{t \in (t_1)_2 \mid t \supseteq q_1'\}\rangle \Vdash z \in y$ and $\langle q_2', \{t \in (t_3)_2 \mid t \supseteq q_2'\} \Vdash z \notin y$. But this contradicts Lemma 5.6. Hence $t_2 \Vdash z \in y$. □

□ of Theorem 5.4.

## 6. What can happen without the axiom of choice

First we return to Specker's $\Omega$ (for the definition of $\Omega$, see the Introduction). In the model $N_G$ of ZF '$\Omega = 0n$' holds by Lemma 3.4. To obtain a model of $\Omega = \omega_{\alpha+1}$ for $\alpha \in 0n$ let $\lambda = \aleph_{\alpha+1}^{N_G}$, i.e. $\lambda$ is the $(\alpha+1)$st cardinal of $N_G$. One can verify using arguments similar to those of Theorem 5.3 (but much simpler) that $N_{G_\lambda}$ is a model of $\Omega = \omega_{\alpha+1}$. Note that to obtain $N_{G_\lambda}$ we need only an initial segment of the sequence $\langle k_\nu \mid \nu \in 0n\rangle$ of all strongly compact cardinals. Thus for the consistency of $\Omega = \omega_{\alpha+1}$ we need the strongly compact cardinals $\langle k_\nu \mid \nu < \alpha\rangle$. In particular, for $\alpha = 2$ we need only one strongly compact cardinal or the weaker assumption that there is cardinal $k$ which is $k^{+n}$-strongly compact for every $n < \omega$. To obtain the consistency of $\Omega = \omega_\Omega$ we start from the assumption that there is a regular cardinal $k$ which is a limit of $< k$-strongly compact cardinals, where $\alpha$ is said to be $< k$-strongly compact if for every $\beta < k$, $\alpha$ is $\beta$-strongly compact. We define $P_1$, $P_2$, $P_3$ only below $k$. The model $N_{G_k}$ thus obtained has the required property. This establishes the following theorem:

THEOREM 6.1. (1) *If* ZFC + '$(\forall \alpha \in 0n)(\exists k > \alpha$ *$k$ is a strongly compact cardinal*$)$' *is consistent then so is* ZF + $\Omega = 0n$.

(2) *If* ZFC + '*there is regular cardinal $k$ which is a limit of $< k$-strongly compact cardinals*' *is consistent then so is* ZF + $\Omega = \omega_\Omega$.

(3) *If* ZFC + *'there are $\alpha$ strongly compact cardinals' is consistent then so is* ZF + $\Omega = \omega_{\alpha+1}$ (*where $\alpha$ is given by an absolute definition*).

DEFINITION 6.2. (Specker [10]) Let $\tilde{G}_0$ = the class of all countable sets, $G'_\alpha$ = the class of all countable unions of sets from $\bigcup\{\tilde{G}_\beta \mid \beta < \alpha\}$ and $\tilde{G}_\alpha = G'_\alpha - \bigcup\{\tilde{G}_\beta \mid \beta < \alpha\}$. Put $\tilde{G} = \bigcup\{\tilde{G}_\alpha \mid \alpha \in 0n\}$. For a set $a \in \tilde{G}$ let grad $a$ be the least $\alpha$ such that $a \in \tilde{G}$.

As is easily seen $V = \tilde{G}$ which asserts that every set has a grad implies $\Omega = 0n$.

THEOREM 6.3. (1) $N_G \models$ *'every set is a countable union of sets of smaller cardinality'*.

(2) $N_G \models V = \tilde{G}$.

PROOF. (1) For a well orderable set the theorem follows from Lemma 3.4. Let now $A \in N_G$ and $N_G \models$ (there is no well ordering of $A$). Let $\boldsymbol{A}$ be a symmetric name of $A$. Then for some $a \subseteq \mathrm{Reg}$ and $\lambda \in \mathrm{Reg}$, $\operatorname{supp} \boldsymbol{A} \subseteq a$ and $\bigcup\{\operatorname{supp} \boldsymbol{x} \mid \boldsymbol{x} \in \operatorname{dom} \boldsymbol{A}\} < \lambda$. Put for $b \in [\lambda]^{<\omega}$ $\operatorname{dom} \boldsymbol{B}_b = \{\boldsymbol{x} \in \operatorname{dom} \boldsymbol{A} \mid \operatorname{supp} \boldsymbol{x} \subseteq b\}$ and $\boldsymbol{B}_b(\boldsymbol{x}) = 1_{\mathrm{RO}(P_3)}$. For every $\pi \in H_a$ and $\boldsymbol{x} \in \operatorname{dom} \boldsymbol{B}_b$, $\pi\boldsymbol{x} \in \operatorname{dom} \boldsymbol{B}_b$. Since $H_b$ is the normal subgroup of the group of permutations, for every $g \in H_b$, $\pi \in H_a$, $\pi^{-1}g\pi$ also is in $H_b$, hence for $\boldsymbol{x} \in \operatorname{dom} \boldsymbol{B}$, $(\pi^{-1}g\pi)\boldsymbol{x} = \boldsymbol{x}$ and $g(\pi\boldsymbol{x}) = \pi\boldsymbol{x}$. So $\operatorname{sym} \boldsymbol{B}_b \supseteq H_a$ and $\{\langle b, \boldsymbol{B}_b\rangle \mid b$ finite subset of $\lambda\} \in N_G$. Find in $M$ a bijection $f : \lambda \leftrightarrow \lambda^{<\omega}$. Then $\{B_{f(\gamma)} \mid \gamma < \lambda\} \in N_G$. Let $\operatorname{dom} \boldsymbol{B}_b = \{\boldsymbol{x}_\nu \mid \nu < \alpha\}$, $\operatorname{sym}\{\langle \check{\nu}, \boldsymbol{x}_\nu\rangle \mid \nu < \alpha\} \supseteq H_b$. Hence in $N_G$ there is well ordering of $B_b$ for $b \in [\lambda]^{<\omega}$.

Put $C_\delta = A \cap \bigcup_{\gamma<\delta} B_{f(\gamma)}$ for $\delta \leqq \lambda$. Find the least ordinal $\eta$ such that $|C_\eta| = |A|$. If $\eta$ is limit ordinal we are done. Suppose that $\eta = \eta_0 + n$ where $\eta_0$ is a limit ordinal and $n < \omega$. $\eta_0 = 0$ is impossible otherwise $A$ is well orderable. Put $C'_\delta = A \cap (C \cup \bigcup_{i \leqq n} B_{f(\eta_0+i)})$. Then $C_\eta = C'_{\eta_0}$. Choose in $N_G$ a sequence $\langle \nu_n \mid n < \omega\rangle$ such that $\bigcup_{n<\omega} \nu_n = \eta_0$. Then $C_\eta = \bigcup\{C'_{\nu_n} \mid n < \omega\}$. If for some $\xi < \eta_0$, $|A| = |C'_\xi|$ then we can do the same with $C'_{\nu_n}$. Notice that since $\xi < \eta_0 < \delta$ and by our assumption $A$ is not well orderable the axion of foundation implies that this process must stop after a finite number of steps.

(2) Show that an $A \in N_G$ has a grad. Let the sequence $\langle C_\delta \mid \delta \leqq \lambda\rangle$ be as in (1). Show by induction that for every $\alpha \leqq \lambda$, $C_\alpha$ has a grad. Suppose that for every $\beta < \alpha$, $C_\beta$ has a grad. By the definition of $\langle C_\delta \mid \delta < \lambda\rangle$, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ if $\alpha$ is a limit ordinal and $C_\alpha = C_{\alpha'} \cup (A \cap B_{f(\alpha')})$ if $\alpha = \alpha' + 1$. In the second case $C_\alpha$ has a grad since $B_{f(\alpha')}$ is a well orderable set in $N_G$ and hence has a grad. For limit $\alpha$ find in $N_G$ an $\omega$-sequence $\langle \beta_n \mid n < \omega\rangle$ with limit $\alpha$. Then $C_\alpha = \bigcup_{n<\omega} C_{\beta_n}$ and

each $C_{\beta_n}$ has a grad. Hence $C_\alpha$ also has a grad. Now $A = \bigcup_{\beta<\delta} C_\beta$ for some $\delta \leqq \lambda$. It follows that $A$ has a grad. □

THEOREM 6.4. *There is a generic extension of $N_G$ in which every ordinal has cofinality $\aleph_0$ but there is an uncountable set $A$ such that for every sequence $\langle A_i \mid i \in I\rangle$ which is increasing with respect to inclusion if $|A_i|<|A|$ for every $i \in I$ and $A = \bigcup_{i\in I} A_i$ then $|I| \geqq |A|$.*

PROOF. Let $C$ be the Cohen's forcing conditions which break the axiom of choice, i.e. $C = \{p \mid p$ is a finite function from $\omega \times \omega \times \omega$ into $2 = \{0,1\}\}$ and $C$ is ordered by inclusion. Permutations on $\langle C, \supseteq\rangle$ are defined as follows. Let $\pi$ be some permutation of $\omega\times\omega$ such that if $\pi(n,m)=(n',m')$ then for all $m$, $n = n'$ or for all $m$, $n \neq n'$, $\pi$ generates permutation on $\langle C, \supseteq\rangle$

$$\operatorname{dom}(\pi p) = \{\langle \pi(n,k), m\rangle \mid \langle n,k,m\rangle \in \operatorname{dom} p\},$$

$$\pi p(\pi(n,k), m) = p(n,k,m),$$

where $p \in C$ and $n, m \in \omega$. Consider the symmetric extension $N'_G$ of $N_G$ which is generated by the set of all $H_l = \{\pi \mid \pi(n,m) = (n,m)$ for $(n,m) \in l\}$ where $l$ is some finite subset of $\omega\times\omega$ (for details see Jech [4], model X11). Let $x_{n,k} = \{m \in \omega \mid (\exists p \in G')\ (p(n,k,m)=1)\}$ where $G'$ is a generic subset of $C$. Let $X_n = \{x_{n,k} \mid k \in \omega\}$. Put $A = \{X_n \mid n \in \omega\}$. By Jech [4] $A$ has the required properties. □

Speker [10] proved that $\operatorname{grad}\omega_\alpha = \alpha$ and $\operatorname{grad}(P(\omega_\alpha)) \geqq \alpha + 1$. Hence the strongest statement in this direction which may hold is:

(K) For every ordinal $\alpha$, $P(\omega_\alpha)$ is a countable union of sets of cardinality $\aleph_\alpha$.

Speker showed that the consistency of (K) implies the consistency of $(V = \tilde{G})$. By means of a proof similar to that of Theorem 6.3 one can show that if (K) is consistent with ZF then (K) does not imply $V = \tilde{G}$ in ZF. By the next theorem $V = \tilde{G}$ does not imply (K) in ZF. But is (K) consistent with ZF? It seems that to prove the consistency of (K) with ZF we need some much stronger assumptions concerning the existence of large cardinals.

THEOREM 6.5. (1) $N_G \models$ ($2^{\aleph_0}$ *is not a countable union of well orderable sets*).
(2) $N_G \models (\operatorname{grad}(2^{\aleph_0}) = 2)$.

PROOF. (1) Suppose that $2^{\aleph_0} = \bigcup\{B_n \mid n < \omega\}$ and each $B_n$ is a well orderable set in $N_G$. Then for every $n < \omega$ there is a finite set $a_n$ of regular cardinals (in $M$) such that for every $x \in B_n$, $\operatorname{sym}\boldsymbol{x} \supseteq H_{a_n}$. One picks $a_n$ as the support of a one to one function from an ordinal onto $B_n$ and easily verifies that for $x \in B_n$, $\operatorname{sym} x \supseteq a_n$. Let $f$ be a symmetric name of the set $\langle B_n \mid n < \omega \rangle$. Define the name $g$ for the sequence $\langle a_n \mid n < \omega \rangle$ as follows:

$$\operatorname{dom} g = \{ \langle \check{n}, \operatorname{s\check{u}pp} a \rangle^{\vee} \mid \langle n, a \rangle \in \operatorname{dom} f \} \quad \text{and} \quad g(\langle \check{n}, \operatorname{s\check{u}pp} a \rangle^{\vee}) = f(\check{n}, a).$$

Since $\operatorname{supp} g \subseteq \operatorname{supp} f$, $g$ is a symmetric name. So $\langle a_n \mid n < \omega \rangle \in N_G$. Find an ordinal $\alpha$ such that $\aleph_0 \leqq \alpha < k_1 = \aleph_1^{N_G}$, $\alpha$ is a regular cardinal in $M$ and there is no $n < \omega$ such that $\alpha \in a_n$. There is such an $\alpha$ since $\bigcup \{ a_n \mid n < \omega \}$ is a countable set in $N_G$. Let $x \in N_G \cap P(\omega \times \omega)$ be a well ordering of $\omega$ by the order type $\alpha$ which is obtained from the generic function $G \mid \{\alpha\}$. Put $y = j''(x)$ where $j : \omega \times \omega \leftrightarrow \omega$. Then $\operatorname{supp} y = \{\alpha\}$ and if $z \in P^{N_G}(\omega)$ is such that $\alpha \not\in \operatorname{supp} z$ then $N_G \models y \neq z$ since $G \mid \{\alpha\}$ is a generic subset of $P_3 \mid \{\alpha\}$ in the model $M[G''(\operatorname{supp} z)] \subseteq N_G$. Thus $y \neq z$ for all $z \in B_n$, $n < \omega$, contradicting $\bigcup_{n<\omega} B_n = P(\omega)$.

(2) For every $b \in [k_1]^{<\omega}$ let $\operatorname{dom} B_b = \{ x \mid \operatorname{dom} x \subseteq \check{\omega} \text{ and } \operatorname{supp} x \subseteq b \}$ and $\operatorname{rng} B_b = \{ 1_{\mathrm{RO}(P_3)} \}$. As in Theorem 6.2, $2^{\aleph_0} \subseteq \bigcup \{ B_b \mid b \in [k_1]^{<\omega} \}$. Set $C_\delta = 2^{\aleph_0} \cap \bigcup \{ B_b \mid b \in [\delta]^{<\omega} \}$ for $\delta \leqq k_1 = \aleph_1^{N_G}$. In $N_G$ each $C_\delta$ for $\delta < k_1$ is a countable union of countable sets. Hence by (1), $2^{\aleph_0} \neq C_\delta$ for $\delta < k_1$. So $2^{\aleph_0} = C_{k_1}$. The cofinality of $k_1$ in $N_G$ is $\aleph_0$. Hence $2^{\aleph_0}$ is a countable union of sets of grad 1. Again (1) implies that $\operatorname{grad}(2^{\aleph_0}) = 2$. □

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